

SOLUTION OF EXERCISES

IN

HALL & STEVEN'S GEOMETRY,

PART III

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SOLUTIONS OF EXERCISES

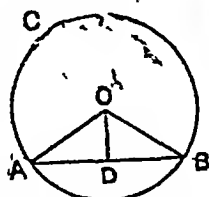
IN

HALL AND STEVEN'S GEOMETRY

PART III.

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1. Draw a line AB $AB=8$ cm. and bisect it at D . At D draw DO perp. to AB making $DO=3$ cm. Join OA and OB .



Now, the $\triangle OAD$ and $\triangle ODB$ are identically equal (Theor 4). $\therefore OA=OB$.

With centre O and radius OA or OB draw the circle ABC . Then ABC is the required circle.

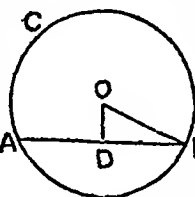
It is required to find the length of OB and verify it by measurement.

From Theor. 29 we have,

$$OB = \sqrt{DB^2 + OD^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5 \text{ cm.}$$

Measure OB and it will be found to be 5 cm. long.

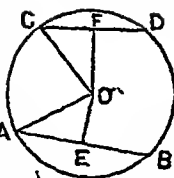
2. Take any point O , and with O as centre and radius $= 13''$ describe a circle ABC . From O draw a st. line, $OD=5''$. Through D draw a st. line ADB perp. to OD , meeting the circumference at A and B . Then AB is the required chord. Join AB .



$$\text{Then from Theor. 29, } DB = \sqrt{OB^2 - OD^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12''.$$

Now, $AB = 2 DB$ (Converse, Theor. 31), $= 2 \times 12$ or $24''$.

3. Take any point O and radius $= 1''$. Take any two points A and C on the circumference. With A as centre and radius $= 1.0''$ and $1.2''$

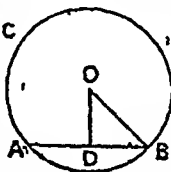


and with O as centre describe a circle $A B C D$. Take any two points A and C on the circumference. With A as centre and radius $= 1.0''$ and $1.2''$ respectively draw arcs cutting the circle at B and D . Join $A B$ and $C D$. Then $A B, C D$ are the required chords. From O draw $O E$ perp. to $A B$, and $O F$ perp. to $C D$. Join $O A$ and $O C$.

Then from Theor. 29, we have $O E = \sqrt{O A^2 - A E^2} = \sqrt{1^2 - 8^2} = \sqrt{36} = 6''$, and $O F = \sqrt{O C^2 - C F^2} = \sqrt{1^2 - 6^2} = \sqrt{64} = 8''$.

Measure $O E$ and $O F$ and it will be found that $O E = 6''$, and $O F = 8''$.

4. Take any pt O and radius $= 1$ cm. Take any point A on the circumference. With A as centre draw an arc cutting

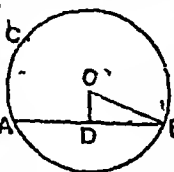


the circle at B . Join $A B$. Then $A B$ is the required chord. From O draw $O D$ perp. to $A B$. Join $A B$.

From Theor. 29, we have $O D = \sqrt{O B^2 - D B^2} = \sqrt{4^2 - 3^2} = \sqrt{7} = 2.6$ cm. approx

Measure $O D$ and it will be found to be 2.6 cm, nearly.

5. With any pt O as centre and radius $= 3.7$ cm, draw a circle. Take any pt. A on the circumference and radius $= 3.7$ cm, draw an arc cutting the circle at

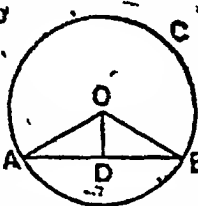


B . Join $A B$. Then $A B$ is the required chord. From O draw $O D$ perp. to $A B$.

From Theor. 29, we have $O D = \sqrt{O B^2 - D B^2} = \sqrt{3.7^2 - 3.5^2} = \sqrt{1.41} = 1.2$ cm.

Measure OD and it will be found to be 1.2 in.
 \therefore the true length of $OD = 12''$ or 1 ft.

6. With any pt. O as centre and radius $= 1.3''$ describe a circle ABC . With any pt. A on the circumference and radius $= 2.4''$ draw an arc cutting the circle at B . Join AB . Then AB is the chord. Join OA, OB . It is required to find the area of the $\triangle AOB$ in sq. in.

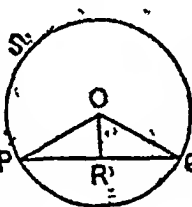


From O draw OD perp. to AB .

From Theor. 29, we have $OD = \sqrt{OB^2 - DB^2} = \sqrt{1.3^2 - 1.2^2} = \sqrt{.25} = .5''$.

Area of the $\triangle AOB = \frac{1}{2} AB \cdot OD = \frac{1}{2} \times 2.4 \times .5 = 6$ sq. in.

7. Let P and Q be two pts. 8'' apart. Join PQ and bisect it at R . At R draw RO perp. to PQ . With P as centre and radius $= 1.7''$ draw an arc cutting RO at O . Join OP and OQ . With centre O and radius $= 1.7''$ draw a circle. This circle will pass through the points P and Q , because, the $\triangle ORP$ and ORQ being identically equal (Theor. 4), $OP = OQ$.



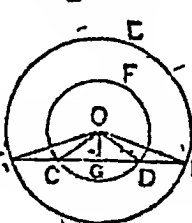
From Theor. 21, we have

$OR = \sqrt{OP^2 - RP^2} = \sqrt{1.7^2 - 1.5^2} = \sqrt{.64} = .8''$.

Measure OR and it will be found to be $.8''$.

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1. Let ABE and CDF be two concentric circles with O as their common centre. Let $ACDB$ be a st. line cutting the two circles at A, B, C and D .

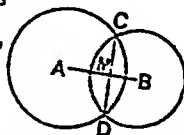


CDF be two concentric circles with O as their common centre. Let $ACDB$ be a st. line cutting the two circles at A, B, C and D .

It is required to prove that the intercepts AC and DB are equal. From O draw OG perp. to AB

Proof.—Then $AG = GB$ and $CG = GD$ (Theor. 31).
 $\therefore AG - CG = GB - GD$, or $AC = DB$ Q. E. D.

2. Let two circles whose centres are at A and B intersect at C , D . Join CD and bisect it at M . Join AM and BM ,



It is required to prove that AM and BM are in the same st. line.

Proof.—Because the st. line AM drawn from the centre A bisects the chord CD

\therefore the $\angle AMD$ is a rt. \angle (Theor 31).

Similarly, the $\angle BMD$ is a rt. \angle (Theor 31)

\therefore the $\angle s$ AMD and BMD together $= 2$ rt. $\angle s$.

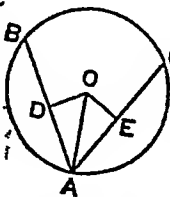
$\therefore AM$ and BM are in the same st. line. (Theor 2)

Hence, it is required to prove that the line of centres bisects the common chord at rt. angles.

Because AB (which is the line of centres) is perp. to CD , and passes through M , the middle point. of CD , (proved above), it bisects the common chord CD at right angles.

Q. E. D.

3. Let AB, AC be any two equal chords, of circle ABC whose centre is O . It is required to show that the st. line which bisects the $\angle BAC$ passes through the centre O .



From O draw OD perp. to AB and OE perp. to AC , Join AO .

Proof—Since OD , OE are perps. to AB , AC respectively,

$\therefore AB$ is bisected at D and AC at E . (Theor. 31, Converse). But $AB = AC$ (given),

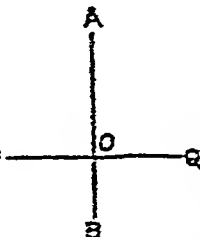
\therefore their halves AD , AE are also equal.

Now, in the $\triangle^s ODA$, OEA ,

$DA = EA$ (proved)
because $\begin{cases} AO \text{ is common to both, and} \\ \angle ODA = \angle OEA, \text{ being rt. } \angle^s \end{cases}$

\therefore two \triangle^s are equal in all respects (Theor. 18), so that the $\angle DAO = \angle OAE$. Hence AO bisects the $\angle BAC$. \therefore the bisector of the $\angle BAC$ passes through the centre O Q. E. D.

4. Let P, Q be any two given points.
It is required to find the locus of the centres of all circles which pass through P and Q . Join PQ and bisect it at O .
Through O draw AOB perp. to PQ .



Proof—Since AOB bisects PQ at right \angle^s ,

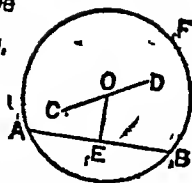
then AOB is the locus of all points equidistant from P and Q (Prob. 14).

Now, the centre of every circle passing through P and Q is a point equidistant from P and Q .

\therefore the locus of the centres of all circles passing through the points P and Q is the st. line AOB which bisects PQ at right angles.

Q. E. D.

5. Let A and B be
and CD a given st.



any two given points
line.

It is required to describe a circle passing through two
points A and B and having its centre on the st. line CD .

Construction — Join AB and bisect it at E . At E
draw EO perp. to AB meeting CD at O .

Since EO bisects AB at rt. \angle at E .

The centre of the circle passing through A and B , lies
on the st. line EO (proved in exercise 4).

The centre also lies on the given st. line CD . (Hypo-
thesis).

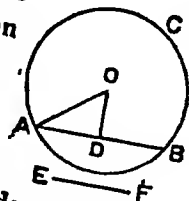
\therefore the pt. O , common to both EO and CD , is the
required centre.

Now with centre O and radius OA or OB describes
the required circle ABF .

Q. E. F.

This problem is impossible when the given st. line
 CD does not meet EO , i. e., is parallel to EO , i. e., is
perp. to the line AB and does not pass through the mid.
pt of AB .

6. Let A and B be
and EF a given



any two given points
st. line.

It is required to describe a circle passing through the
points A and B and having a radius $= EF$.

Construction — Join AB and bisect it at D . At D
draw DO perp. to AB . With centre B and radius $= EF$,
draw an arc cutting DO at O .

Since OD bisects AB at rt. \angle^s ,

the centre of the required circle passing through A and B lies on the st. line DO (proved in exercise 4). And since OA is equal to the st. line EF (by construction),

$\therefore O$ is the centre of the required circle.

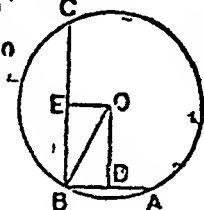
Now, with centre O and radius OA describe the required circle $A'B'C$.

Q. E. F.

This problem is impossible when the given st. line EF is less than AD , i. e., less than half the st. line AB ; for then the arc drawn with B as centre would not cut DO , and the construction would fail.

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1. Let $AB = 1.5''$ and $BC = 5''$ be two st. lines at rt. \angle^s to each other.



It is required to draw a circle passing through the points A , B and C , to find the length of the radius of the circle and to verify it by measurement.

The locus of centres of the circles passing through the points C and B is the st. line EO which bisects CB at rt. \angle^s at E . Similarly, the locus of centres of the circles passing through the points B and A is the st. line DO bisecting BA at rt. \angle^s at D .

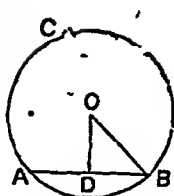
\therefore the point O , common to both EO and DO , is the centre of the required circle, passing through A , B and C .

Now, with centre O and radius OB draw a circle. It will pass through C and A also. Join OB .

$$\text{Radius } OB = \sqrt{OE^2 + EB^2} = \sqrt{BD^2 + EB^2} = \sqrt{.8^2 + 1.5^2} = \sqrt{2.89} = 1.7''$$

Measure OB and it will be found to be $1.7''$.

2. Draw a st. line
sect it at D. At D
A B, making D O =



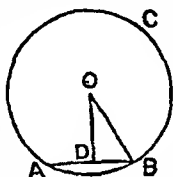
A B = 6 cm. and bi-
draw D O perp. to
3 cm.

With centre O and radius = O A or O B draw the
circle A B C Join O-B

Radius O B = $\sqrt{O D^2 + D B^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.2$
cm. nearly.

Measure O B and it will be found to be 4.2 cm.

3. With any point O
= 4 cm. draw the
any point B on the
centre B and radius
cutting the circle



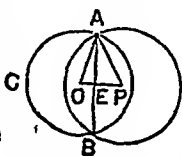
C as centre and radius
circle A B C. Take
circumference. With
= 4 cm. draw an arc
cutting the circle at A Join A B.

Then A B is the required chord. Join O B From O draw
O D perp to A B

O D = $\sqrt{O B^2 - D B^2} = \sqrt{4^2 - 2^2} = \sqrt{12} = 3.5$ cm.
nearly.

Measure O D and it will be found to be 3.46 cm.

4. With any point O as centre and radius = 2.5 cm. de-
scribe the circle A B C the circumference
centre A and radius C D = 4.8 cm. draw an
arc cutting the circle at B Join A B. From



Take any pt. A on
of the circle. With
= 4.8 cm. draw an
at B Join A B. From

O draw O E perp to A B, then O E will bisect A B at E
(converse, Theor 31) With centre B and radius = 2.6 cm.
draw an arc cutting O E produced at P.

With centre P and radius = 2.6 cm draw a circle.
Then it will pass through the points A and B. Join A C
and A P.

result by measurement.

$$OE = \sqrt{AO^2 - AE^2} = \sqrt{2.5^2 - 2.4^2} = \sqrt{.49} = 7 \text{ cm.}$$

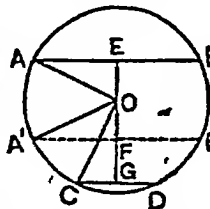
$$EP = \sqrt{AP^2 - AE^2} = \sqrt{2.6^2 - 2.4^2} = \sqrt{1.00} = 1.0 \text{ cm.}$$

$$\therefore OP = OE + EP = 17 \text{ cm.}$$

Measure O P and it will be found to be 1.7 cm.

\therefore the true distance between the centres of the circles = 17"

5. With O as centre and radius = $6.5''$ draw the circle A C D B. Take any pt. A on the circumference. - With centre A and radius = $12''$ draw an arc cutting the circle at B. Join A B. From O draw O E perp. to A B.



From E A and E B cut off lengths each = 2.5". From those pts draw perps. to AB cutting the circle at C and D. Join CD. Then CD is parallel to AB and is equal to 5". Produce EO to meet CD in G. Then OG is also perp. to CD (Theor 14). Join OA and OC. From OG cut off OF = OE. Through F draw A'F'B' parallel to AB. Then A'B' = AB. Join O A'.

It is required to show that the distance between C D and A B or A'B' is 8 5" or 3 5."

$$OG = \sqrt{OC^2 - CG^2} = \sqrt{6.5^2 - 2.5^2} = \sqrt{36} = 6''.$$

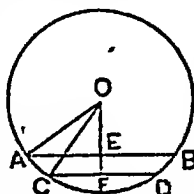
$$OE = \sqrt{OA^2 - AE^2} = \sqrt{6.5^2 - 6^2} = \sqrt{6.25} = 2.5''$$

$\therefore EG$ (the distance bet. AB and CD) $= EO + OG$
 $= 6 + 2.5$ or $8.5''$.

And FG (the distance between $A'B'$ and CD)
 $= OG - OF = OG - OE = 6 - 2.5 = 3.5$.

6. Draw a st. line $AB=8$ cm. and bisect it at E At E draw EF perp. to AB making $EE'=1$ cm.

Through F draw DE making $FC = 3$ cm. $CD = 6$ cm. Join AC and bisect it at right angles by a line meeting FE at E . FC parallel to AB and $FD = 3$ cm. Then AC and bisect it at right angles by a line meeting FE at E .



produced at O. Then O is the centre of the circle. With centre O and radius OA draw the circle ACDB.

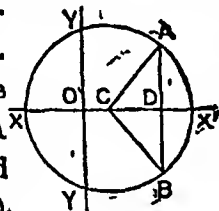
Join OA and OC. Let OE = x cm. Then OF = OE + EF = $(x+1)$ cm.

Now $OA^2 = OC^2$ (being radii), or $OE^2 + AE^2 = OF^2 + CF^2$ (Theor. 29), or $x^2 + 4^2 = (x+1)^2 + 3^2$,
 $\therefore x = 3$ cm.

\therefore The radius OA = $\sqrt{x^2 + 4^2} = \sqrt{3^2 + 4^2} = 5$ cm.

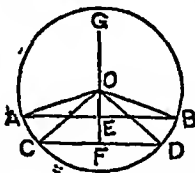
Measure OA and it will be found to be 5 cm.

7. Plot the pts. A and B whose co-ordinates are (5,5) and (6,-5) respectively. Join AB. Then DA and DB are each = 5. Take any point C on the x -axis. Join CA and CB. Now, the $\Delta^s CDA$ and CDB are identically equal (Theor. 4). $\therefore CA = CB$.



Hence, the circle drawn with centre O and passing through A must also pass through B.

8. Let AB, CD be any two parallel chords in the circle ACDB whose centre is O. Bisect AB at E and CD at F. Join OE and OF.



Proof—Now OE is perp. to AB, and OF is perp. to CD. (Theor. 24).

Since AB and CD are parallel, OF is also perp. to AB. Now from O two perps OE and OF are drawn to AB. Hence these lines must coincide, i.e., O, E and F must be on the same st line OEF.

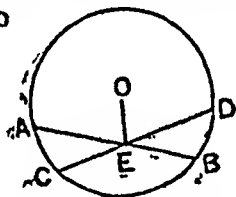
Q. E. D.

9. See Fig. in Ex. 8.—Let CD be any chord of the circle $ACDB$ whose centre is O . Through O draw GOF perp. to CD cutting CD at F . Then F the mid. point of CD .

Proof.—Draw any chord AB parallel to CD , cutting FO at E . Then OE is perp. to AB . $\therefore E$ is the mid. point of AB (Theor. 31, converse). And E lies on FG . Similarly it can be shown that the middle point of any other chord drawn parallel to CD lies on FG . Hence FG is the required locus.

Q. E. D.

10. Let $ACBD$ be a circle whose centre is O , and let AB, CD be two chords intersecting at E .



It is required to prove that the chords AB, CD cannot bisect each other unless each is a diameter.

If possible let the chords AB, CD bisect each other at E . Join OE .

Proof—Since E is the middle point of AB , therefore $\angle OEB$ is a rt. \angle (Theor. 31).

Again since E is the middle point of CD , the $\angle OED$ is a rt. \angle (Theor. 31).

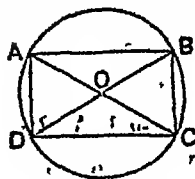
\therefore The $\angle OEB = \angle OED$.

The part is equal to the whole, which is absurd.

Hence, AB, CD cannot bisect each other. But if each be a diameter, they would intersect at centre O . And obviously a diameter is bisected at the centre.

Q. E. D.

11. Let $A B C D$ be a parallelogram inscribed in a circle and let the diagonals $A C, B D$ intersect at O .



It is required to prove that O is at the centre of the circle.

Proof.—Since the diagonals $A C, B D$ of the parallelogram bisect one another (Cor. 3, Theor. 21) at O , and each is a chord of the circle. Hence each must be a diameter (proved in Ex 10).

$\therefore O$ where the diagonals intersect is at the centre of the circle

12 See Fig in Ex. 11.—Let $A B C D$ be a parallelogram inscribed in a circle and let the diagonals $A C, B D$ intersect each other at O .

It is required to show that the parallelogram $A B C D$ must be a rectangle.

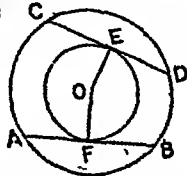
Proof—Each of the diagonals $A C, B D$ must be a diameter (proved in Ex 11), and hence they are equal.

\therefore the parallelogram $A B C D$ is a rectangle (Ex. 5; page 58).

Q. E. D.

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✓ 1. Let $A B, C D$ be any two of a system of equal chords of a circle whose centre is O , and F and E be their mid. points.



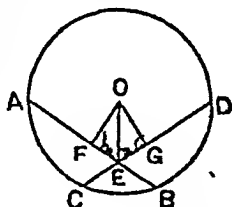
It is required to find the locus of the point E or F .
Join OE and OF .

Proof—Since equal chords of a circle are equidistant from the centre, therefore $OF = OE$ (Theor. 34).

\therefore the middle point of any one of the given system of equal chords is at a distance $= OF$ from the centre.

\therefore the required locus is a circle whose centre is O and radius $= OF$, the common distance of the equal chords from the centre O .

2. Let AB, CD , the two chords of a circle whose centre is O , cut one another at E , such that the $\angle AEO = \angle OED$.



It is required to prove that AB, CD are equal.

From O draw OF perp. to AB and OG perp. to CD .

Proof—In the $\triangle^s OFE$ and OEG

because $\begin{cases} \text{the } \angle OFE = \text{the } \angle OEG \text{ (given)} \\ \text{the } \angle OFE = \text{the } \angle OGE, \text{ being rt. } \angle^s \\ \text{and } OE \text{ is common to both} \end{cases}$

\therefore the two \triangle^s are equal in all respects (Theor. 17);
so that $OF = OG$.

$\therefore AB = CD$ (converse, Theor. 34).

Q. E. D.

3. See fig. in Ex. 2.—Let the two equal chords AB, CD of a circle whose centre is O , intersect at E .

It is required to prove that $AE = ED$, and $EB = CE$.
From O draw OF , perp. to AB , and OG perp., to CD .
Join OE .

Proof—Because $AB = CD$, $\therefore OF = OG$ (Theor. 34).

In the $\triangle^s OFE$ and OEG ,

because $\begin{cases} OF = OG \\ OE \text{ is common to both} \\ \text{and the } \angle OFE = \text{the } \angle OGE, \text{ being rt. } \angle^s \end{cases}$

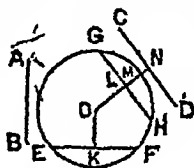
\therefore the two \triangle 's are congruent (Theor. 18); so that $FE = EG$.

Because OF is perp. to AB ; $\therefore F$ is middle point of AB . (converse, Theor. 31):

For the same reason, G is the middle point of CD .

Now $AB = CD$ (given). $\therefore AF = FB = CG = GD$.
 $\therefore AF + FE = GD + EG$, i. e. $AE = ED$. Also $FB = FE = CG - EG$; i. e. $EB = CE$. Q. E. D.

4. Let O be the centre of the given circle, AB and CD two given st. lines of which AB is not greater than the diameter of the circle,



It is required to draw a chord in the given circle which shall be equal to AB , and parallel to CD .

Construction.—Take any point E on the circumference of the circle. With centre E and radius $= AB$, draw an arc cutting the circle at F . Join EF . From O draw OK perp. to EF , and ON perp. to CD . From ON cut off $OL = OK$. Through L draw HLG perp. to ON meeting the circle at G and H . Then GH is the required chord.

Proof.—Since $OL = OK$ (by construction),

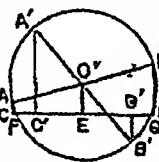
$\therefore GH = EF$ (converse, Theor. 34) $= AB$.

Again, since GH and CD are perps to ON ,

$\therefore GH$ and CD are parallel (Ex. 2, page 41).

Q. E. F.

5. Let PQ be a fixed chord of the circle whose centre is O ; let AB and $A'B'$ be any two diameters, of which the latter cuts the chord PQ while the former does not. Draw $AC, BD, A'C', B'D'$ perps. to PQ , meeting PQ produced or PQ at C, D, C', D' .



It is required to prove that the sum of the perps. AC and DB , and the difference of the perps. $A'C'$ and $B'D'$ are constant for all positions of A, B .

From O draw $O E$ perp. to $P Q$.

Proof— $O E = \frac{1}{2} (AC + BD)$ or $\frac{1}{2} (A'C' - B'D')$,

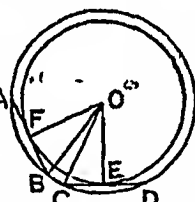
since A and B are on the same side, and A' and B' on opposite sides of $P Q$ (Ex. 9, page 65).

Since the chord $P Q$ is fixed (given), therefore $O E$ its distance from the centre O is of constant length.

Hence, $(AC + BD)$ or $(A'C' - B'D')$ is constant.

Q. E. D.

6. With any point O as centre and radius $= 4.1$ cm. draw the circle $A'B$ $C D$. With any pt. A on the circumference, as centre and radius $= 1.8$ cm. draw an arc cutting the circle $A'B$ as centre and radius $= 1.8$ cm. draw an arc cutting the circle $A'B$ at B . Join $A B$. Then $A B$ is the reqd. chord. Similarly draw the chord $C D = 1.8$ cm. Bisect $A B$ at F and $C D$ at E . Join $O F$, $O B$, $O C$ and $O E$.



Because $O F$ bisects the chord $A B$, therefore it cuts $A B$ at rt. \angle^s (Theor. 31). And $O F = \sqrt{O B^2 - B F^2} = \sqrt{4.1^2 - 1.8^2} = 4$ cm.

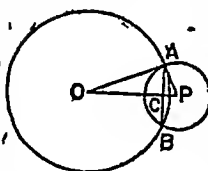
Similarly $O E$ cuts $C D$ at rt. \angle^s (Theor. 31); and $O E = \sqrt{O C^2 - C E^2} = \sqrt{4.1^2 - 1.8^2} = 4$ cm. $\therefore O F = O E$.

\therefore The points F, E , as well as the middle points of all chords 1.8 cm. long, lie on a circle whose centre is O and radius $= 4$ cm.

Measure $O E$ and it will be found to be 4 cm.

With centre O and radius $= 4$ cm. draw the circle.

7 With any point O as centre and radius $= 3.7''$ draw a circle. Take any pt. A on the circumference of the circle. With centre A and radius $= 2.4''$ draw an arc cutting the circle at B . Then A and B are reqd. pts. Join AB . From O draw OC perp. to AB , then AB is bisected at C (converse, Theor. 31) Produce OC to P making $OP = 4''$. Then P is the centre of the smaller circle.



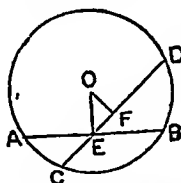
With centre P and radius $= PA$ draw a circle. This circle passes through the point B also Join OA and AP .

$$OC = \sqrt{OA^2 - AC^2} = \sqrt{3.7^2 - 1.2^2} = 3.5''. \therefore CP = OP - OC = 4 - 3.5 = .5''.$$

$$\therefore PA \text{ the radius of smaller circle} = \sqrt{AC^2 + CP^2} = \sqrt{1.2^2 + .5^2} = 1.3''.$$

Page 153.

1. Let ACB whose centre is O , given point in it.



be the given circle and let E be the

It is required to draw the least possible chord through E .

Join OE . Through E draw AB perp. to OE meeting the circle at A and B . Then AB is the reqd. chord.

Let CD be any other chord through E . Draw OF perp. to CD .

Then in the rt. angled $\triangle EFO$, OE (being the hypotenuse) is greater than OF .

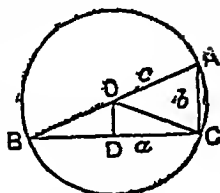
$\therefore CD$ is greater than AB (Theor. 35)

Similarly it can be proved that every other chord through the point E is greater than AB .

Hence AB is the least possible chord that can be drawn through E

Q. E. F.

2. Take a st. line $BC = 3.5''$. With centres B and C, and radii equal to $3.7''$ and $1.2''$ respectively draw two arcs cutting one another at A. Join AB and AC. Then $\triangle ABC$ is the required



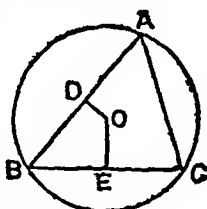
Now $a^2 + b^2 = 3.5^2 + 1.2^2 = 13.69 = 3.7^2 = c^2$, \therefore the triangle is rt. \angle at C.

Construction—Bisect BC at D. At D draw DO perp. to BC meeting BA in O. Then O is the centre of the reqd. circle. Join OC. With centre O and radius OC draw the circle ABC which passes through A and B also, since O is the mid. pt. of AB (Ex. 10, page 47.)

\therefore BA is the diameter of the circle ABC, \therefore radius $= \frac{1}{2} BA = \frac{1}{2} \times 3.7$ or $1.85''$.

Measure OC and it will be found to be $1.85''$.

3. Construct the $\triangle ABC$ such that $AB = 3''$, $BC = 2.8''$. It is reqd. to draw the circum-circle of the $\triangle ABC$ and measure its radius.



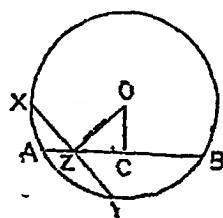
$\triangle ABC$ such that $AB = 3''$, $BC = 2.8''$ and $AC = 2.6''$. the circum-circle to measure its radius.

Construction—Bisect AB at D and BC at E. At D draw DO perp. to AB; at E draw EO perp. to BC meeting DO in O. Then O is the centre of the reqd. circle (Theor. 32).

With centre O and radius OA draw the circle ABC.

Measure OA and it will be found to be $1.62''$.

4. Let O be the centre of the circle of which AB is the fixed chord. Take any pt. Z in AB. Join OZ. Through Z draw the chord XZY perp. to OZ.



centre of the circle fixed chord. Take Join OZ. Through Z draw the chord XZY perp. to OZ.

Then the chord XY has its middle pt Z (Converse, Theor 31) on AB . From O draw OC perp. to AB , then AB is bisected at C (Theor. 31).

It is reqd. to find the greatest and the least length that XY may have.

Proof.—The length of XY depends upon its distance from the centre O , i. e., on OZ (Theor 31). XY will be greatest for least value of OZ , and least for greatest value of OZ .

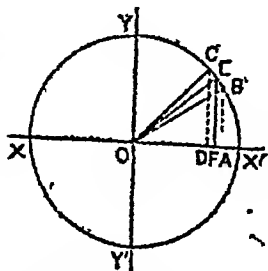
Now since Z is any pt. on AB , OZ will be least when it coincides with OC , the perp. from O to AB (Theor 12). In that case XY becomes the chord AB . Hence AB is the greatest length of XY .

Again, since Z must be on AB , OZ is greatest when OZ coincides with OA or OB . In that case the length of the chord XY becomes zero, which is its least value.

Again as Z approaches from A or B to C (the foot of the perp.) length of OZ diminishes (Cor. 3, Theor. 12). $\therefore XY$ increases as Z approaches C , the mid. pt. of AB .

Q. E. D.

5. Plot the
ordinates are ($2.4''$,
 C whose co-or-
 $2.4''$).



pt. B whose co-or-
 $1.8''$), also the pt.
ordinates are ($1.8''$,

With the origin O as centre and radius $= 3''$ describe a circle.

Join CB and bisect it at E . From E draw EF perp. to XX' . Join OE . Draw CD , BA perps to XX' .

Because $OB = \sqrt{OA^2 + BA^2} = \sqrt{2.4^2 + 1.8^2} = \sqrt{9.00} = 3''$, and $OC = \sqrt{OD^2 + CD^2} = \sqrt{1.8^2 + 2.4^2} = \sqrt{9.00} = 3''$,

$\therefore OB = OC = 3''$, = the radius.

Hence, the pts. B and C are on the circle.

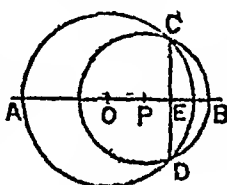
(i) From B draw a perp. to CD, and suppose it cuts CD at F. Then $CB = \sqrt{CF^2 + FB^2}$; but $FB = DA = OA = OD = 2.4 - 1.8$ or $.6''$; and $CF = CD - BA = 2.4 - 1.8$ or $.6''$. $\therefore CB = \sqrt{.6^2 + .6^2} = \sqrt{.72} = .848'' = .85''$ approx.

(ii) $OF = \frac{1}{2}(OA + OD) = \frac{1}{2}(2.4 + 1.8) = 2.1''$; and $EF = \frac{1}{2}(CD + BA) = \frac{1}{2}(1.8 + 2.4) = 2.1''$.

(iii) OE (perp. from O) = $\sqrt{OF^2 + EF^2} = \sqrt{2.1^2 + 2.1^2} = \sqrt{8.82} = 2.969'' = 2.97$ approx.

Page 155.

1. Let AB be a given st line, and C a given pt. It is all circles, whose centres lie on AB and which pass through C, must pass through a second fixed point.



given st line, and reqd. to prove that centres lie on AB through the fixed through a second

Draw CE perp. to AB. Produce CE to D making $ED = CE$. Then D is the second fixed pt.

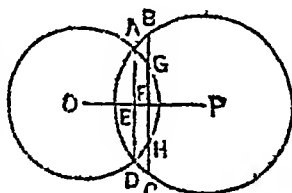
Proof—Since AB bisects CD at rt. angles,

\therefore all the pts. on AB are equidistant from C and D (Prob. 14),

\therefore the circles whose centres, O, P, etc., lie on AB and which pass through C, also pass through D.

Q. E. D.

2 Let two circles ABCD whose centres are O and P intersect at A and D. Then AD is the common chord. Let a line BC parallel to AD cut these circles at B, G, H and C.



circles AGHD and centres are O and A and D. Join is the common st. line BC parallel these circles at B,

It is reqd. to prove that the intercepts BG and HC are equal.

Join OP cutting AD at E and BC at F .

Proof—Since OP bisects AD at rt. \angle^s (Ex. 2, page 147) and BC is parallel to AD

$\therefore OP$ cuts BC at rt. \angle^s (Ex. 3, page 41);

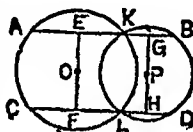
and since BC is the chord of the circle $ABCD$ it bisects BC (Converse, Theor. 31), i.e., $BF = FC$,

Again since GH is the chord of the circle $AGHD$ and OF is perp. to it, OF bisects GH (Converse, Theor. 31), i.e., $GF = FH$.

$\therefore BF - FG = FC - FH$, or $BG = HC$.

Q. E. D.

3. Let two circles DB whose centres are another at the pts. K and L , Let AKB and CLD be two parallel lines drawn through K and L , cutting the circles at A, B, C, D .



$AKLC$ and KL O and P cut one and L , Let AKB let so lines drawn ting the circles at

It is reqd. to prove that $AB = CD$.

Through O draw FOE perp to AK and CL . Through P draw HPG perp. to KB and LD .

Proof— EF and GH are parallel (Ex. 2, page 41), and AB, CD are parallel, therefore the figure $EFGH$ is a parallelogram.

$\therefore EG = FH$ (Theor. 21)

Since OE is perp. to AK , therefore OE bisects AK at E (Converse, Theor. 31), so that $EK = \frac{1}{2} AK$,

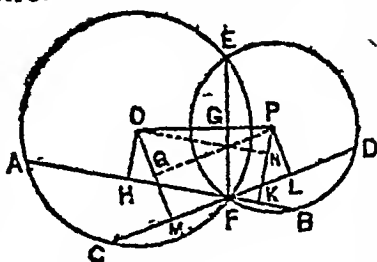
Similarly, $KG = \frac{1}{2} KB$, $FL = \frac{1}{2} CL$, and $LH = \frac{1}{2} LD$.

$\therefore EK + KG = \frac{1}{2} (AK + KB)$, and $FL + LH = \frac{1}{2} (CL + LD)$, i.e., $EG = \frac{1}{2} AB$, and $FH = \frac{1}{2} CD$,

But $EG = FH$ (proved), therefore $AB = CD$.

Q. E. D.

4. Let two circles $ACFE$ and $EFBD$ whose centres are O and P meet at E and F . then EF is the common chord. Through F draw AFB and CFD making equal angles with EF (*i.e.*, the $\angle AFE = \angle EFB$ and the $\angle EFD = \angle CFE$) and terminated by the circumferences at A, B, C and D .



It is reqd. to prove that AB and CD are equal.

From O draw OH, OM perps. to AF, CF respectively; from P draw PK, PL perps. to FB, FD respectively. From O draw ON perp. to PK , and from P draw PQ perp. to OM . Join OP cutting EF at G .

Proof— OP bisects EF at rt. angles (Ex. 2, page 147).

Now, in the quadrilateral $OMFG$, the $\angle^s OMF$ and OGF are rt. angles; therefore the $\angle^s MOG$ and MFG are supplementary. Similarly, in the quadrilateral $GFKP$, the $\angle^s GFK$ and GPK are supplementary.

But the $\angle MFG = \angle GFK$ (given), therefore the $\angle MOG = \angle GPK$.

Now, in the $\triangle^s OQP$ and ONP

because $\begin{cases} \text{the } \angle QOP = \angle OPN \text{ (proved)} \\ \text{the } \angle OQP = \angle ONP, \text{ being rt. } \angle^s \\ \text{and } OP \text{ is common to both} \end{cases}$

\therefore two \triangle^s are equal in all respects (Theor. 17); so that $QP = ON$.

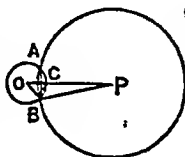
The figure $OHKN$ is a parallelogram; $\therefore ON = HK$ (Theor. 21). Since OH is perp. to AF and PK perp. to FB , therefore OH bisects AF and PK bisects FB (Converse, Theor. 31), that is, HF is $\frac{1}{2} AF$, and FK $\frac{1}{2} FB$. $\therefore HF + FK = \frac{1}{2} (AF + FB)$, or $HK = \frac{1}{2} AB$. $\therefore ON = \frac{1}{2} AB$.

Similarly, it can be proved that $QP = \frac{1}{2} CD$.

But $QP = ON$ (proved), therefore $AB = CD$.

Q. E. D.

5. Draw a st. line sect AB at C . Through W with centre B and radius = 3.7 cm. draw an arc cutting CO at O and CP at P . With radii equal to 2 cm. and 3.7 cm. respectively draw two circles



$AB = 2.4$ cm. BC draw perp. $OC \perp$ CP $OC = 2$ cm. draw O . With centre B draw another arc centres O and P respectively draw

It is reqd. to find the length of OP and verify it by measurement. Join OB and BP .

$$OC = \sqrt{OB^2 - BC^2} = \sqrt{2^2 - 1.2^2} = \sqrt{2.56} = 1.6 \text{ cm.}$$

$$\text{And } CP = \sqrt{BP^2 - BC^2} = \sqrt{3.7^2 - 1.2^2} = \sqrt{12.25} = 3.5 \text{ cm.}$$

$$\therefore OP = OC + CP = 1.6 + 3.5 = 5.1 \text{ cm.}$$

Measure OP and it will be found to be 5.1 cm.

\therefore True length of $OP = 5.1$ cm.

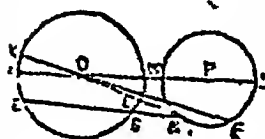
6. See fig. in Ex. 5 — Make a st. line $OP = 2.1$ cm. With centres O and P , and radii equal to 1 cm and 1.7 cm respectively draw two circles, intersecting at A and B . Join AB cutting OP at C . Then AB is the common chord.

It is reqd. to find by calculation, and by measurement, the length of AB , and the lengths of OC and CP .

Join OB and BP .

Let $OC = x$, then $CP = OP - OC = 2.1 - x$. Now, $OB^2 - OC^2 = CB^2 = BP^2 - CP^2$, or $1^2 - x^2 = 1.7^2 - (2.1 - x)^2$, or $4.2x = 2.52$, $\therefore x = .6$ cm, i. e., $OC = .6$ cm. $\therefore CP = 2.1 - .6 = 1.5$ cm. $\therefore CB = \sqrt{OB^2 - OC^2} = \sqrt{1^2 - .6^2} = \sqrt{.64} = .8$ cm. $\therefore AB = 2 \times .8 = 1.6$ cm.

1. Let O and P be the centres of two given circles $AEGB$ and $CHFD$ which do not intersect. Join OP cutting the circles at B and C . Produce OP both ways to again cut the circles at A and D . Let any other st. line $EGHF$ cut the circles at E, G, H and F . Join FO and produce it to meet the circumference at K .



It is reqd. to prove that (i) AD is the greatest, and (ii) BC the least of the st. lines which have one extremity on each of two given circles.

Proof—(i) Since from the external pt. O , the st. line OD is drawn through the centre P , and OF is any other line,

$\therefore OD$ is greater than OF (Theor. 37). To these unequals add equals OA and OK .

$\therefore OD + AO$ are greater than $OF + KO$,
i.e. AD is greater than KF .

Again, since from the external pt. F , the st. line FK is drawn through the centre O , and EF is any other line,

$\therefore KF$ is greater than EF (Theor. 37).

$\therefore AD$ is much more greater than EF .

Similarly, it can be proved that AD is greater than any other line having one extremity on each of the two circles.

Hence AD is the greatest of all such lines.

(ii) Join HO cutting the circle $AEGB$ at L .

Because, HL when produced passes through the centre O , and HG does not, and they are drawn from the external pt. H ,

$\therefore HL$ is less than HG (Theor. 37).

Again since OC when produced passes through the centre P , and OH does not, and they are drawn from the external pt. O ,

$\therefore OC$ is less than OH . And since $OE = OL$,

$\therefore OC - OB$ is less than $OH - OL$,

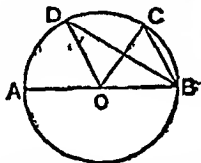
i.e., BC is less than LH ; but LH is less than GH (proved). $\therefore BC$ is much more less than GH .

Similarly, by taking any number of st. lines terminated by the circumference of the circles, it can be proved that BC is less than them all.

Hence BC is the least of all such lines.

Q. E. D.

2. Let $ABCD$ be a circle whose centre is O , and from any pt B on the circumference the lines BOA , BD to the circumference $\angle BOD$ subtended by BD at the centre O greater than the $\angle BOC$ subtended by BC .



It is reqd. to prove that of these st. lines, (1) BA is the greatest, and (2) BD is greater than BC .

Join OD , OC .

Proof.—(1) In the $\triangle BOD$, the sides BO , OD are together greater than BD (Theor. 11).

But $OD = OA$, being radii;

$\therefore BO$, OA are together than BD ,

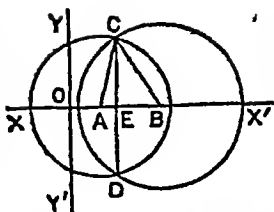
i.e., BA is greater than BD .

Similarly it can be proved that BA is greater than any other straight line drawn from B to the circumference.

Hence, BA is the *greatest* of all such lines.

(ii) In the two $\triangle^s DOB$ and COB ,

4. Take any two pts. A and B on the x -axis. Let D be the pt. whose co-ordinates are $(8, -11)$. With centres A and B , and radii AD, BD respectively draw two circles intersecting at the pt. C .



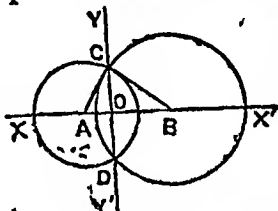
two pts. A and B . Let D be the pt. whose co-ordinates are $(8, -11)$. and B , and radii AD, BD respectively draw two circles intersecting again at the

It is reqd. to find the co-ordinates of C .

Join CD . Then CD is bisected at rt. angles by AB (Ex. 2, page 147), i.e., by the x -axis.

Hence, the co-ordinates of C are $(8, 11)$.

5. Plot the pts. A, B and C whose co-ordinates are $(-6, 0), (15, 0)$ and $(0, 8)$ respectively. With centres A and B and radii AC, BC respectively draw two circles intersecting again at D .



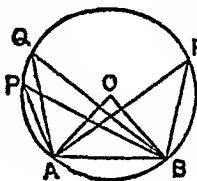
It is reqd. to find the lengths of the radii of two circles, and the co-ordinates of the pt. D .

Join AC and CB .

Because both the centres A and B lie on the axis of x , and pt. C lies on the y -axis, the st. line CD is bisected at rt. angles at the origin O by the x -axis. Therefore, the co-ordinates of the pt. D are $(0, -8)$.

$$\therefore AC = \sqrt{CO^2 + AO^2} = \sqrt{8^2 + 6^2} = 10; \text{ and } CB = \sqrt{CO^2 + OB^2} = \sqrt{8^2 + 15^2} = 17.$$

6 Let OAB be an isosceles triangle with an angle of 80° at its vertex O . With centre O and radius OA draw a circle. Let P, Q, R, \dots be any number of pts. on the circumference of the circle on the same side of AB as the centre O . Join $AP, BP, AQ, BQ, AR, BR, \dots$



It is reqd. to measure the angles $\angle APB$, $\angle AQB$, $\angle ARB$, ... subtended by the chord AB at the pts. P , Q , R ,...

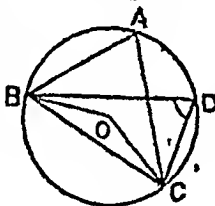
Measure the $\angle^s APB$, AQB , ARB , and it will be found that each of them is equal to 40° .

Now, make the $\angle AOB = 50^\circ$, and repeat the same exercise. It will be found that each of the $\angle^s APB$, AQB is equal to 25° .

Inference—The angles at circumference of a circle subtended by any chord are all equal to one another; and each of them is half of the angle at the centre subtended by the chord.

Page 161.

1. Let $\angle BAC$, $\angle BDC$ be angles in the same segment (major) $BADC$ of a circle, whose centre is O . Join OB , OC . The angle $\angle BDC$ is given 74° . It is reqd. to find the number of degrees in each of the $\angle^s BAC, BOC, OBC$.



The $\angle BAC =$ the $\angle BDC$ (Theor. 39)
 $= 74^\circ$,

the $\angle BOC = 2$ the $\angle BDC$ (Theor. 38), $= 2 \times 74 = 148^\circ$.

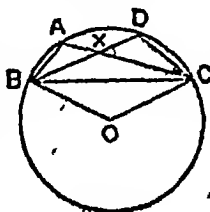
[Since $OB = OC$, being radii, \therefore the $\angle OBC =$ the $\angle OCB$ (Theor. 5).]

But the $\angle^s BOC$, OBC and OCB together $= 180^\circ$ (Theor. 16)

$$\therefore \angle OBC + \angle OCB = 180^\circ - \angle BOC$$

$$\text{or, } 2 \angle OBC = 180^\circ - 148^\circ = 32^\circ. \therefore \angle OBC = 16^\circ.$$

2. Let $\angle BAC$ be (minor) segment whose centre is O . BD and CA intersect at X . The $\angle DXC$ is given 40° , 25° .



angles in the same segment $BADC$ of a circle. Join OB , OC . Let BD and CA intersect at X . The \angle and the $\angle XCD$,

It is reqd. to find the number of degrees in the \angle BAC and in the reflex \angle BOC.

In the $\triangle DXC$, the $\angle^s XDC, DXC$ and XCD together $= 180^\circ$ (Theor. 16),

$$\therefore \angle XDC = 180^\circ - (40^\circ + 25^\circ) = 115^\circ.$$

But the \angle BAC = the \angle BDC (Theor. 39), $= 115^\circ$.

$$\begin{aligned} \text{The reflex } \angle \text{ BOC} &= 2 \text{ the } \angle \text{ BDC (Theor. 38),} \\ &= 2 \times 115^\circ = 230^\circ \end{aligned}$$

✓ 3. See fig. in Ex. 1 — The \angle CBD is given 43° , and the \angle BCD, 82° .

It is reqd. to find the number of degrees in the \angle^s ABC, OBD, OCD.

In the $\triangle DBC$, the $\angle^s BDC, DBC$ and BCD together $= 180^\circ$ (Theor. 16), and the \angle CBD $= 43^\circ$, and the \angle BCD $= 82^\circ$,

$$\therefore \text{the } \angle \text{ BDC} = 180^\circ - (43^\circ + 82^\circ) = 55^\circ.$$

$$\therefore \text{the } \angle \text{ BAC} = \text{the } \angle \text{ BDC (Theor. 39),} = 55^\circ.$$

$$\therefore \text{the } \angle \text{ BOC} = 2 \text{ the } \angle \text{ BDC (Theor. 38)} = 2 \times 55^\circ = 110^\circ$$

Since $OB = OC$, being radii; therefore the \angle OBC = the \angle OCB (Theor. 5). In the $\triangle OBC$, the \angle^s BOC, OBC, OCB together $= 180^\circ$ (Theor. 16), and the \angle BOC $= 110^\circ$. $\therefore \angle \text{ OBC} + \angle \text{ OCB} = 180^\circ - 110^\circ$, or $2 \angle \text{ OBC} = 70^\circ$. $\therefore \angle \text{ OBC} = 35^\circ = \angle \text{ OCB}$

$$\begin{aligned} \angle \text{ OBD} &= \angle \text{ DBC} - \angle \text{ OBC} = 43^\circ - 35^\circ = 8^\circ; \text{ and } \angle \\ \text{ OCD} &= \angle \text{ BCD} - \angle \text{ OCB} = 82^\circ - 35^\circ = 47^\circ. \end{aligned}$$

✓ 4. See fig in Ex 2. — It is reqd. to show that the \angle OBC = \angle BAC $- 90^\circ$.

Proof — In the \triangle BOC, because the \angle^s BOC, OBC and OCB together $= 180^\circ$, (Theor. 16), and the \angle OBC = the \angle OCB (Theor. 5), $\therefore 2 \angle \text{ OBC} = 180^\circ - \angle \text{ BOC} = 180^\circ - (360^\circ - \text{reflex } \angle \text{ BOC}) = \text{reflex } \angle \text{ BOC} - 180^\circ$.

$$\therefore \angle \text{ OBC} = \frac{1}{2} \text{ reflex } \angle \text{ BOC} - 90^\circ.$$

But the $\angle BAC = \frac{1}{2}$ reflex $\angle BOC$ (Theor. 38)

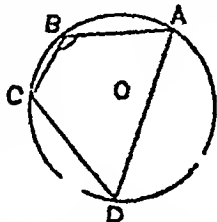
$\therefore \angle OBC = \angle BAC - 90^\circ$.

Q. E. D.

Page 163.

1. With any pt. O as centre and radius = 1.6 draw the circle $ABCD$.

A on the circum-
ference. At B make the \angle
 BC meeting the cir-
cumference at C .
Take any pt. D on
 B . Join DC and
is the reqd. inscrib-



Take two pts. B and
ference. Join BA .
 $ABC = 126^\circ$, the arm
cumference at C .
the arc opposite to
 DA . Then $ABCD$
ed quadrilateral.

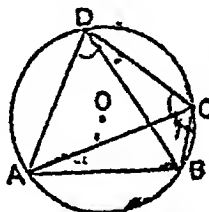
Measure the $\angle^s BCD, CDA$, and BAD , and it will
be found that the $\angle BCD = 114^\circ$, the $\angle CDA = 54^\circ$, and
the $\angle BAD = 66^\circ$.

(Note—The $\angle ADC$ will always be equal to 54° ; but
the $\angle^s BCD$ and BAD may have different values de-
pending on the position of D .)

The $\angle^s ABC$ and $ADC = 126^\circ + 54^\circ = 180^\circ$; and the
 $\angle^s BCD$ and $BAD = 114^\circ + 66^\circ = 180^\circ$.

Hence, the opposite angles of the inscribed quadrila-
teral $ABCD$ are supplementary.

2. Let $ABCD$ be
inscribed in the circle
DB.



a quadrilateral
 ABC . Join AC ,

It is reqd. to prove by the aid of Theorems 39 and
15, that the $\angle^s ADC, ABC$ together $= 2$ rt. $\angle^s =$ the \angle^s
 BAD, BCD together.

Proof—Since the $\angle ADB =$ the $\angle ACB$, and the \angle
 $BDC =$ the $\angle BAC$ (Theor. 39).

\therefore the $\angle ADC =$ the $\angle ADB +$ the $\angle BDC =$ the \angle
 $ACB +$ the $\angle BAC$.

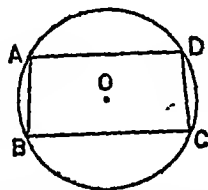
To these equals add the $\angle ABC$.

Then, the $\angle ADC + \text{the } \angle ABC = \text{the } \angle^s ACB + BAC + ABC = 2 \text{ rt } \angle^s$ (Theor. 16)

Similarly it can be proved that the $\angle^s BAD, BCD$ together $= 2 \text{ rt. } \angle^s$.

Q. E. D.

3. Let $ABCD$ be about which a circle It is reqd to prove gram. $ABCD$ is a



a parallelogram can be described that the parallelo rectangle.

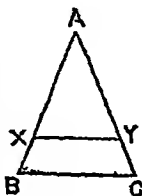
Proof—Because $ABCD$ is a cyclic quadrilateral, therefore the opp. $\angle^s BAD$ and BCD together $= 2 \text{ rt. } \angle^s$ (Theor. 40)

But the $\angle BAD = \text{the opp. } \angle BCD$ (Theor. 21).

\therefore each of the $\angle^s BAD$ and BCD is a $\text{rt. } \angle$, and since quadrilateral $ABCD$ is a parallelogram, it is a rectangle.

Q. E. D.

4. Let ABC be and let XY be drawn BC , cutting the sides It is reqd. to prove Y lie on a circle.



an isosceles triangle, parallel to the base AB, AC in X and Y . the four pts. $B, C, X,$

Proof—Since $AB = AC$ (given), therefore the $\angle ABC = \text{the } \angle ACB$ (Theor. 5).

Since XY and BC are parallel and XB meets them,

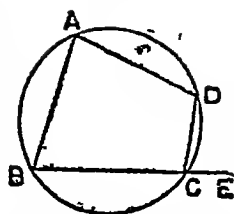
\therefore the $\angle^s YXB$ and XBC together $= 2 \text{ rt. } \angle^s$ (Theor. 14)

\therefore the $\angle^s YXB$ and YCB together $= 2 \text{ rt. } \angle^s$.

Hence, the pts. B, C, X, Y are concyclic (Converse, Theor. 40).

Q. E. D.

5. Let $ABCD$ be a cyclic quadrilateral and let BC be produced to E . It is reqd. to prove that the exterior $\angle DCE =$ the interior $\angle BAD$.



be a cyclic quadrilateral. It is reqd. to prove that the exterior $\angle DCE =$ the interior $\angle BAD$.

Proof—Because $ABCD$ is a cyclic quadrilateral, therefore the $\angle BAD$ is supplement of the $\angle BCD$ (Theor. 40).

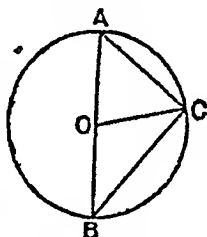
Also, the $\angle ECD$ is supplement of the $\angle BCD$ (Theor. 1)

\therefore the $\angle BAD =$ the $\angle DCE$ [Cor. 3, (2), Theor. 1].

Q. E. D.

Page 165.

1. Let ABC be a triangle rt. angled at C . It is reqd. to prove that the circle described on the hypotenuse AB as a diameter passes through the angular pt C . Bisect AB at O .



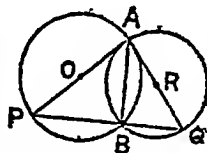
a triangle rt. angled at C . It is reqd. to prove that the circle described on the hypotenuse AB as a diameter passes through the angular pt C . Join OC .

Proof—Since $OC = \frac{1}{2} AB$ (Ex. 10, page 47), therefore $OC = OA = OB$.

Hence, a circle described with centre O and radius OB will pass through the pts. A and C .

Q. E. D.

2. Let two circles whose centres are O and R intersect at A and B . Let two lines APQ and BQR be drawn through A and B respectively.



APB and AQB are straight lines, and R is the intersection of diameters AP , AQ .

It is reqd. to prove that the pts. P, B, Q are collinear. Join AB, PB, BQ .

Proof—Since AP is a diameter of the circle APB , therefore the $\angle ABP$ is a rt. \angle (Theor. 41).

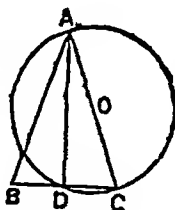
Again since AQ is a diameter of the circle AQB , the $\angle ABQ$ is a rt. \angle (Theor. 41).

\therefore the \angle^s ABP and ABQ together = 2 rt. \angle^s

Hence, PB, BQ are in the same st. line, i.e., the pts. P, B, Q are collinear.

Q. E. D.

3. Let ABC be an isosceles triangle and on one of the equal sides AC as a diameter let the circle cutting BC at D.



an isosceles triangle equal sides AC as a circle ACD be described

It is reqd. to prove that D is the middle pt. of BC. Join AD.

Proof—Since AC is the diameter of the circle ACD,

\therefore the $\angle ADC$ is a rt. \angle (Theor. 41),

$\therefore \angle ADB$ is also a rt. \angle .

Now in the \triangle^s ABD and ADC,

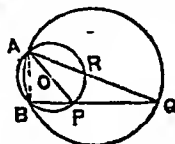
because $\begin{cases} AB=AC \text{ (given)} \\ AD \text{ is common to both} \\ \text{and the } \angle ADB = \text{the } \angle ADC, \text{ being rt. } \angle^s, \end{cases}$

\therefore two \triangle^s are equal in all respects (Theor. 18)

so that $BD=DC$, i. e. D is the mid. pt. of BC.

Q. E. D.

4. Also see fig. in Ex. 2.—Let APQ be a triangle. Let two circles APB and ABQ be described on AP, AQ as diameters, and let them intersect again at B.



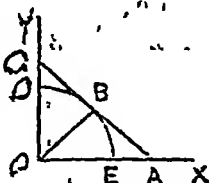
It is reqd. to prove that the point B lies on the third side PQ, or PQ produced. Join AB.

Proof—Since AP is a diameter, the $\angle ABP$ is a right angle [Theor. 41]. For the same reason the $\angle ABQ$ is a right angle.

And these angles have one arm AB common, \therefore the other arms BP and BQ must lie in the same str. line, since at B there can be only one perp. to AB ; i.e., P, B and Q lie on the same str. line, i.e., B lies on PQ or PQ produced.

Q. E. D.

✓ 5. Let AC denote the straight rod, sliding straight rulers OX and OY placed at right angles to one an-



one position of between two OY placed at other.

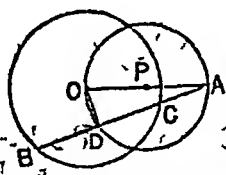
It is reqd. to find the locus of the middle point of the rod AC . Bisect CA at B . Join OB .

Proof—Then $OB = \frac{1}{2} AC$ (Ex. 10, page 47). The length of AC is constant, therefore the length of OB is also constant. And since O is a fixed point, the locus of B is a circle whose centre is O and radius $OB = \frac{1}{2} AC$.

But since the rod AC slides between the rulers OX and OY , its middle pt. B never goes beyond these rulers. Therefore, the reqd. locus is the arc DBE .

Q. E. D.

✓ 6. Let O be the centre of the given circle and A any pt. outside it. It is reqd. to find the locus of the middle pts. of chords of the given circle drawn through



centre of the given circle and A any pt. outside it. It is reqd. to find the locus of chords of the given circle drawn through the fixed pt. A .

From A draw a str. line ACB cutting the circle at C and B . Then CB is a chord through A . From O draw OD perp. to BC . Join OA .

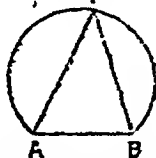
Since OD is perp. to BC , therefore OD bisects BC at D . (converse, theor. 31.)

Since ODA is a rt. angled triangle, rt. angled at D , \therefore the circle ODA described upon the hypotenuse OA as diameter passes through D .

Similarly it can be proved that the middle pts. of all chords drawn through A lie on the circle ODA . And since the mid. pt. of a chord must lie within the circle the locus of the mid. pts. of all chords drawn through A , is an arc of the circle ODA described upon OA as diameter, enclosed by the given circle. The same reasoning can be applied when A is on, or within the circumference of the given circle. OA is less than, equal to or greater than the radius of the given circle, according as the pt. A lies within, on, or without the circumference of the given circle; also, in the last case when A lies without, the locus is only an arc, while in the other two cases the locus is the complete circle.

Pages 170-1

1. Let P be any pt. on the arc of a segment of which AB is the chord. join PA, PB .



It is reqd. to show that the sum of the \angle^s PAB, PBA is constant.

Proof—In the $\triangle PAB$, the sum of the \angle^s APB, PAB and PBA

$= 180^\circ$ (Theor. 16)

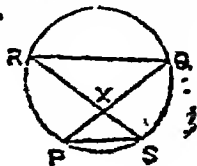
\therefore the $\angle PAB +$ the $\angle PBA = 180^\circ -$ the $\angle APB$.

But the $\angle APB$ is constant (Theor. 39)

Hence, the sum of the \angle^s PAB, PBA is constant.

Q. E. D.

2 Let PQ, RS be two chords of a circle intersecting at X . Join RQ, PS .



It is reqd. to prove that the \triangle^s PXS and RXQ are equiangular to one another.

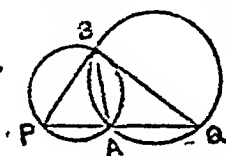
Proof.—The $\angle RQP =$ the $\angle RSP$, also, the $\angle QRS =$ the $\angle QPS$, (Theor. 39)

And, the $\angle RXQ =$ the $\angle PXS$ (Theor. 3)

\therefore the \triangle^s PXS and RXQ are equiangular to one another.

Q. E. D.

3. Let two circles intersect at A and B , and through A let any st. line PAQ by drawn terminated by the circumferences. Join PB, BQ .



It is reqd. to show that PQ subtends a constant angle at B i.e., the $\angle PBQ$ is constant. Join BA .

Proof.—In the $\triangle PBQ$, the sum of the \angle^s PBQ, BPQ and $PQB = 180^\circ$ (Theor. 16.)

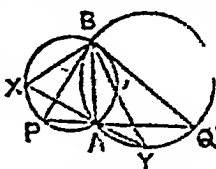
\therefore the $\angle PBQ = 180^\circ - (\angle BPQ + \angle PQB)$

Since the chord AB is fixed, the angles in the segments APB and AQB are of constant magnitudes. (Theor. 39.)

\therefore the sum of the \angle^s BPA and BQA is constant, or the $\angle PBQ$ is constant.

Q. E. D.

4. Let two circles intersect at A and B , and through A let any two st. lines PAQ, XAY be drawn terminated by the circumferences.



Join PB, BQ, XB, BY .

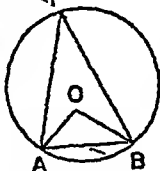
It is reqd. to show that the arcs PX, QY subtend equal angles at B , i.e., the $\angle XBP =$ the $\angle YBQ$.

Proof—The $\angle PBX =$ the $\angle PAX$, being in the same segment $PABX$ (Theor. 39), for the same reason the $\angle YBQ =$ the $\angle YAQ$.

But the $\angle PAX =$ the $\angle YAQ$ (Theor. 3); $\therefore \angle PBX = \angle YBQ$.

Q. E. D.

5. Let P be any pt. on the arc of a segment whose chord is AB , and let the \angle^s PAB , PBA , be bisected by st. lines which intersect at O . It is reqd. to find the locus of the pt. O .



Proof—In the $\triangle PAB$, $\angle APB + \angle PAB + \angle ABP = 180^\circ$ (Theor. 16),

$\therefore \frac{1}{2}\angle APB + \frac{1}{2}\angle PAB + \frac{1}{2}\angle ABP = 90^\circ$, or, $\frac{1}{2}\angle PAB + \frac{1}{2}\angle PBA = 90^\circ - \frac{1}{2}\angle APB$, or $\angle OAB + \angle OBA = 90^\circ - \frac{1}{2}\angle APB$

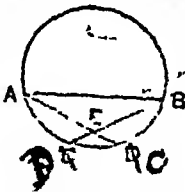
Again, in the $\triangle OAB$, the $\angle AOB + \angle OAB + \angle ABO = 180^\circ$ (Theor. 16.)

$\therefore \angle AOB + 90^\circ - \frac{1}{2}\angle APB = 180^\circ$, $\therefore \angle AOB = 180^\circ - (90^\circ - \frac{1}{2}\angle APB)$ $\therefore \angle AOB = 90^\circ + \frac{1}{2}\angle APB = \text{constant}$, (Theor. 39).

Since, $\angle APB$ is constant.

Hence the locus of the pt. O is an arc of a segment on the fixed chord AB , and containing an angle $= 90^\circ + \frac{1}{2}\angle APB$ (Converse, Theor. 39). Q. E. D.

6. Let two chords AC , DB intersect within the circle at E . It is reqd. to prove that the $\angle AED$ or $\angle BEC =$ the angle subtended by half the sum of the arcs AD and BC . Join AB .



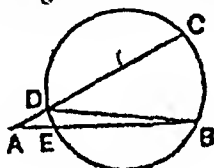
Proof—The angle at the circumference subtended by an arc = twice the angle at the circumference subtended by half the arc = the angle at the centre subtended by half the arc.

\therefore the angle at the centre subtended by half the sum of the arcs AD and BC = the sum of the angles at the circumference subtended by the arcs AD and BC = the sum of the \angle^s ABD, and BAC = the ext. \angle AED (Theor. 16).

Similarly it can be proved that the \angle AEB = the angle at the centre subtended by half the sum of the arcs AB and DC.

Q. E. D.

7. Let two chords CD, BE intersect outside the circle at A.



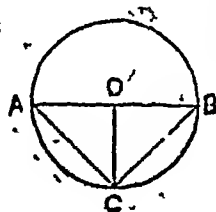
It is reqd. to prove that the \angle CAB = the angle at the centre subtended by half the difference of the arcs BC and DE. Join DB.

Proof—Since the angle at the centre subtended by half an arc = the angle of the circumference subtended by that whole arc (proved in Ex. 6)

\therefore the angle at the centre subtended by half the difference of the arcs BC and DE = the difference of the angles at the circumference subtended by the arcs BC and DE = the difference of the \angle^s BDC and DBE = the \angle BAC, because the ext. \angle BDC = \angle BAC + \angle DBA (Theor. 16); and $\therefore \angle$ BDC - \angle DBA = \angle BAC.

Q. E. D.

8. Let AC, CB be two chords intersecting at right angles.



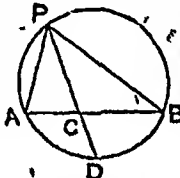
two chords intersecting at right angles.

It is reqd. to prove that the sum of the arcs cut off by AC and CB = the semi-circumference.

Proof—It has been proved in Ex. 6, that the angles at the centre subtended by half the sum of the arcs by the chords = angle bet the chords = 90° , \therefore the angle at the centre subtended by the sum of the arcs = $2 \times 90^\circ = 180^\circ$. \therefore the sum of the arcs = semi circumference since a semi-circumference only can subtend an angle = 180° at the centre.

Note—If the chords do not intersect the proposition does not hold. Q. E. D.

9. Let AB be a fixed chord of a circle and P any pt. on the arc APB. Join PA, PB. Let PD, the bisector of the $\angle APB$ meet the conjugate arc ADB at D. It is reqd. to prove that for all position of P, D is a fixed pt.



Proof—Since the $\angle APD = \angle DPB$ (given), therefore the arc AD = the arc DB (Theor. 42).

\therefore D is the mid. pt. of the arc ADB and hence it is a fixed pt. Q. E. D.

10. Let AB, AC be any two chords. Bisect the arc AB at B and the arc AC at Q. Join BQ cutting AB at X and AC at Y. It is reqd. to prove that AX = AY. Join PB, PA, AQ, QC.



Proof—Since arc AP = arc PB, and arc AQ = arc QC, therefore the $\angle PAB = \angle PBA$, and the $\angle QAC = \angle QCA$ (Theor. 43).

The $\angle APQ = \angle ACQ$, (Theor. 39) = the $\angle QAC$.

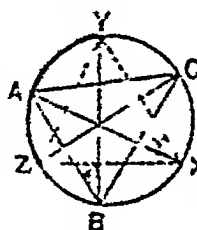
Also, the $\angle PQA = \angle PBA$, (There. 39), = the $\angle PAB$.

Now the ext. $\angle AXY = \angle^s APQ + PAB = \angle^s APQ + PQA$; also the ext. $\angle AYX = \angle^s PQA + QAC = \angle PQA + APQ$ (Theor. 16).

\therefore the $\angle AXY =$ the $\angle AYX$; hence $AX = AY$.

Q. E. D.

11. Let ABC be a triangle inscribed in a circle. Let the bisectors of the $\angle^s BAC, ABC$ and ACB meet the circumference at X, Y and Z . Join XY, YZ and ZX . It is reqd. to prove that, the $\angle YXZ = 90^\circ - \frac{1}{2} A$, B and the $\angle YZX = 90^\circ - \frac{1}{2} C$.



Proof—The $\angle ZXA =$ the $\angle ZCA$, and the $\angle AXY =$ the $\angle ABY$ (Theor. 39).

$$\begin{aligned} \text{The } \angle YXZ &= \angle AXY + \angle ZXA \\ &= \angle ABY + \angle ZCA = \frac{1}{2} B + \frac{1}{2} C. \end{aligned}$$

In the $\triangle ABC$, the sum of the $\angle^s A, B, C = 180^\circ$ (Theor. 16)

$$\therefore \frac{1}{2} A + \frac{1}{2} B + \frac{1}{2} C = 90^\circ$$

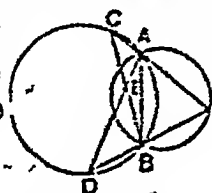
$$\therefore \frac{1}{2} B + \frac{1}{2} C = 90^\circ - \frac{1}{2} A$$

$$\therefore \angle YXZ = 90^\circ - \frac{1}{2} A.$$

Similarly it can be proved that the $\angle ZYX = 90^\circ - \frac{1}{2} B$, and the $\angle YZX = 90^\circ - \frac{1}{2} C$.

Q. E. D.

12. Let two circles ACD and ABP intersect at A and B , and let P be any pt. on the circle ABP . Join PA, PB and produce them to meet the circle ACD at C and D .



It is reqd. to prove that the arc CD is of constant length for all positions of P . Join AB, AD and CB .

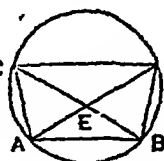
Proof—Since the chord AB is fixed, \therefore the segments $ACDB$ and APB are constant, and $\therefore \angle^s APB$ and ACB are constant (Theor. 39).

Now, the ext. $\angle DBC =$ the $\angle APB +$ the $\angle ACB$
(Theor. 16) = constant.

Hence the arc CD is constant (Converse, Theor. 39).

Q. E. D.

13. Let CD and
chords of a circle C
 CB, DA, DB It is
 $CA = DB$, and $CB =$



AB two parallel
 $CABD$. Join CA ,
reqd. to prove that
 AD .

Proof—Since the $\angle DCB =$ the $\angle CBA$ (Theor. 14),
therefore the minor arc $BD =$ the minor arc CA (Theor.
42),

\therefore the chord $DB =$ the chord CA (Theor. 45.)

The $\angle CDB$ is supplement of the $\angle CAB$ (Theor. 40)
and the $\angle CDB$ is supplement of the $\angle ABD$ (Theor. 14)
 \therefore the $\angle CAB =$ the $\angle ABD$.

\therefore the arc $CB =$ the arc AD (Theor. 42)

\therefore the chord $CB =$ the chord AD (Theor. 45).

Q. E. D.

14. Let two equal circles XAP, AQY intersect one
another at A

st. lines PAQ ,
terminated by the



Through A let two
 XAY be drawn
circumferences.

Join XP and YQ .

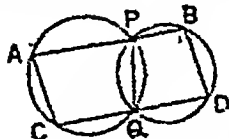
It is reqd. to prove that chord $PX =$ chord QY .

Proof—Because the $\angle XAP =$ the $\angle QAY$ (Theor. 3),
 \therefore the arc $XP =$ the arc QY (Theor. 42),

\therefore the chord $XP =$ the chord QY (Theor. 45)

Q. E. D.

15. Let two circles intersect at P and Q. Through P and Q let two parallel st. lines APB and CQD be drawn terminated by the circumferences. Join AC, BD.



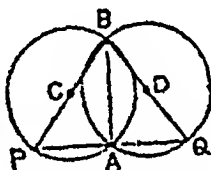
It is reqd. to prove that $AC = BD$. Join PQ.

Proof—Because AP is parallel to CQ, and PB is parallel to QD (given), therefore $AC = PQ$, and $PQ = BD$ (Ex. 13)

$\therefore AC = BD$.

Q. E. D.

16. Let two equal ABQ intersect at A through A let any st. line PAQ be drawn terminated by the circumferences. Join BP and BQ.



circles PBA and ABQ intersect at A through A let any st. line PAQ be drawn terminated by the circumferences.

It is reqd. to prove that $BP = BQ$. Join BA.

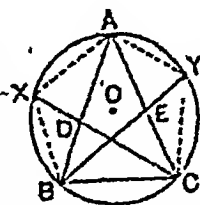
Proof—Since the circles PBA and ABQ are equal, and the chord BA is common to both,

\therefore the minor arc BDA = the minor arc ACB (Theor. 44). $\therefore \angle BPA = \angle BQA$.

$\therefore BP = BQ$ (Theor. 6).

Q. E. D.

17. Let ABC be inscribed in the circles the bisectors of the angles A and B meet the circumference at X and Y. Join AX, AY, XC, YC.



an isosceles triangle AXBCY, and let base angles ACB and BCB be equal. Join AX, AY, XC, YC.

It is reqd. to prove that the four sides BX, XA, AY and YC of the figure BXCAY are equal.

Proof—The $\angle ABC = \angle ACB$ (Theor. 5). \therefore their halves are equal to one another.

\therefore the \angle^s ABY , YBC , ACX and XCB are equal to one another.

\therefore the arcs on which these angles stand are also equal (Theor. 42)

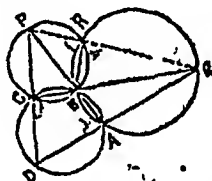
\therefore the chords which cut off these arcs are also equal (Theor. 45)

That is, the chords AY , YC , AX and XB are equal to one another.

Q. E. D.

In order that the figure $BXAYC$ be equilateral, the side BC must be equal to BX , \therefore arc BC must = arc BX (Theor. 44) $\therefore \angle BAC$ must = $\angle BCX$ (Theor. 43), = $\frac{1}{2} \angle ACB$ = half the base angle.

18. Let $AECD$ be a cyclic quadrilateral; and let AB , DC be produced CB , DA to meet at P , and the circles circumscribed PBC , QAB intersect again at R . Join PR and RQ .



be a cyclic quadrilateral; and let the opp. sides to meet at P , and Q . Let the circles circumscribed about the \triangle^s PBC and QAB intersect again at R .

It is reqd. to prove that the pts. P , R , Q are collinear. Join BR .

Proof—Since the quadrilateral $BCPR$ is concyclic, $\angle PRB$ is supplement to the $\angle PCB$ (Theor. 40). But $\angle DCB$ is supplement to the $\angle PCB$. $\therefore \angle PRB = \angle DCB$.

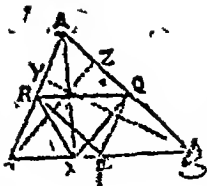
Again, since the quadrilateral $ABRQ$ is concyclic, the $\angle BRQ$ is supplement to the $\angle BAQ$. (Theor. 40). But $\angle BAD$ is supplement to the $\angle BAQ$. $\therefore \angle BRQ = \angle BAD$.

Now the \angle^s $BCD + BAD = 2$ rt. \angle^s . (Theor. 40). \therefore the \angle^s $PRB + BRQ = 2$ rt. \angle^s .

\therefore PR and RQ are in the same st. line (Theor. 2), i.e., the pts. P , R , Q are collinear.

Q. E. D.

19. Let ABC be a triangle and let P, Q, R be the mid. pts. of BC, AB and AC respectively. Let X be the foot of the perp. from the vertex A on the opp. side BC . It is reqd. to prove that the four pts. P, Q, R, X



a triangle and let P, Q, R be the mid. pts. of BC, AB and AC respectively. Let X be the foot of the perp. from the vertex A on the opp. side BC . It is reqd. to prove that the four pts. P, Q, R, X

Join QP, QR, RP, QX and RX .

Proof—Since AXB is a rt. angled triangle, and Q is the mid. pt. of the hypotenuse AB , therefore $QX = QA$ (Prob. 10)

\therefore the $\angle QXA =$ the $\angle QAX$ (Theor 5).

Similarly, in the rt. $\triangle AXC$, the $\angle RXA =$ the $\angle RAX$. $\therefore \angle QXA + \angle RXA = \angle QAX + \angle RAX$, that is, the whole $\angle QXR =$ the whole $\angle QAR$,

Again since $AQPR$ is a parallelogram (Ex. 2, page 64), the $\angle QPR =$ the $\angle QAR$ (Theor. 21)

\therefore the $\angle QXR =$ the $\angle QPR$

\therefore the pts. P, Q, R, X are concyclic (Converse, Theor. 39)

Q. E. D.

20. See figure in Ex. 19.—Let ABC be a triangle and let P, Q, R be the middle pts. of BC, AB and AC respectively. Let X, Y, Z be the feet of the perps. from the vertices A, B, C on opp. sides BC, AC and AB respectively.

It is reqd. to prove that Z, Q, P, X, R, Y are concyclic. Join QP, QR, RP, QX and RX .

Proof—It has been proved in Ex. 19 that the pts. P, Q, R, X are concyclic, i. e., the circle through P, Q, R , also passes through X .

Similarly it can be proved that the circle through Q, P, R passes through Z and also through Y .

But only one circle can pass through the pts. P, Q and R . (Theor. 32),

\therefore the pts. Z, Q, P, X, R, Y are concyclic.

Hence the mid. pts. of the sides of a triangle and the feet of the perps. let fall from the vertices on opp. sides are concyclic.

Q. E. D.

21. Let PAQ, PBQ be a series of triangles standing on the fixed base PQ and having their vertical $\angle^s PAQ, PBQ =$ a given angle. Let the bisectors of the vertical $\angle^s PAQ, PBQ$ meet in C . It is reqd. to prove that C is a fixed point.



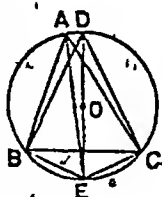
Proof—Since the base PQ is fixed and the $\angle PAQ =$ the $\angle PBQ$,

\therefore the vertices A, B, \dots of the $\triangle^s PAQ, PBQ, \dots$ lie on the arc $PABQ$ of the circle $APQB$ of which PQ is the chord (Converse, Theor. 39.)

\therefore the bisectors of the vertical angle shall in all positions of A pass through C , the mid. pt. of the minor arc, PQ (Ex. 9, page 170).

Q. E. D.

22. Let ABC be in a circle, and let E , the arc BEC subtense remote from A . diameter ED be



a triangle inscribed, be the mid. pt. of ded by BC on the Through E let the drawn.

It is reqd. to prove that the $\angle DEA = \frac{1}{2} (\angle ABC + \angle ACB)$ Join BD, BE, DC, CE .

Proof—Because the arc $BE =$ the arc EC (given), therefore the $\angle BDE =$ the $\angle EDC$ (Theor. 43).

Since DE is a diameter, therefore the $\angle^s DBE$ and DCE are rt. \angle^s (Theor. 41).

Now, in the \triangle^s DBE, DCE, the \angle BDE = the \angle EDC, and, the \angle DBE = the \angle DCE (proved), therefore the \angle BED = the \angle DEC (Theor. 16, Inference 2).

The \angle DEC = \angle AEC - \angle AED = \angle BEA + \angle AED.

$\therefore 2 \angle$ AED = \angle AEC - \angle BEA.

But the \angle AEC = the \angle ABC and the \angle BEA = the \angle ACB (Theor. 39),

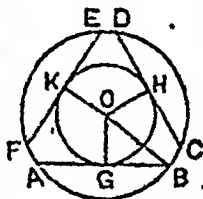
$\therefore 2 \angle$ AED = \angle ABC - \angle ACB.

$\therefore \angle$ AED = $\frac{1}{2} (\angle$ ABC - \angle ACB).

Q. E. D.

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1. With any pt. O as centre and radii = 5 cm. and 3 cm. draw two concentric circles ABC and GHK. Draw a series of chords of the inner circle GHK at right angles to the radii OG, OH, OK respectively. Join OG, OH, OK, then these are perps. to AB, CD, EF respectively (Theor. 46).



Because OG, OH and OK are equal to one another being radii of the same circle \therefore AB, CD and EF are equal to one another (Converse Theor. 34).

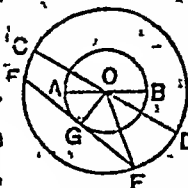
Join OB. Then $GB = \sqrt{OB^2 - OG^2} = \sqrt{5^2 - 3^2} = 4$ cm. But $AB = 2 GB$ (Converse Theor. 31) = 2×4 or 8 cm. = length of each chord of the system. On measurement each will be found to be 8 cm. long.

2. See figure in Ex. 1. — With any pt. O as centre and radius = 1" draw the circle ABC. Make the chords AB, CD, EF, each = 1.6". From O draw OG, OH, OK perps. to AB, CD, EF respectively.

Since $AB = CD = EF$, therefore $OG = OH = OK$ (Theor. 34). Hence these chords touch the concentric circle GHK whose radius is OG. Join OB.

Radius $OG = \sqrt{OB^2 - GB^2} = \sqrt{1^2 - .8^2} = \sqrt{.36} = .6$ "

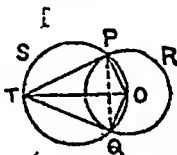
3. With any pt. O as centre and radii = 5 cm. and 2.5 cm. draw two concentric circles CED and AGB. Draw the diameters CD and AB of the two concentric circles. Draw any chord FE of the circle AGB at G. Join OG and OE.



$GE = \sqrt{OE^2 - OG^2} = \sqrt{5^2 - 2.5^2} = \sqrt{18.75} = 4.33$ cm. nearly.

But $EF = 2 GE$ (Converse, Theor. 31) $= 2 \times 4.33 = 8.7$ cm. nearly.

4. Since TSO is upon TO as a diameter, $\angle TPO$ is a rt. \angle



the circle described meter, therefore the (Theor. 41)

\therefore the tangent $TP = \sqrt{TO^2 - OP^2} = \sqrt{13^2 - 5^2} = 12$.

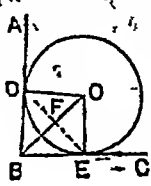
Make the st. line $TO = 5.2$ cm. With centre O and radius = 2 cm. draw a circle. On TO as diameter draw the circle TSO cutting the former circle at P and Q. Join TP, TQ, PO and QO. Then TP and TQ are the two tangents.

The $\angle TOP =$ the $\angle TOQ$ (Cor. Theor. 47). Measure the $\angle TOP$ and it will be found to be 67° .

5. See figure in Ex 4.—With any pt. O as centre and radius = 7" draw a circle. Take any radius OP. At P draw the tangent $PT = 2.4$ ". With centre T and radius TP draw an arc cutting the circle again at Q. Join TQ. Then TQ is the other tangent. Join TO.

$TO = \sqrt{TP^2 + PO^2} = \sqrt{2.4^2 + 7^2} = \sqrt{6.25} = 2.5$ "

6. Let AB, BC be intersecting at A, and let circle touching the lines BO. It is reqd. to bisect the $\angle ABC$.



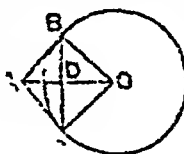
two st. lines, intersecting at A. O be the centre of a circle touching the lines AB at D and BC at E. Join OD, OE. prove that BO bisects the $\angle ABC$.

Proof—In the \triangle^s DBO and OBE , because $OD=OE$ (being radii), BO is common to both, and $BD=BE$ (Theor. 47, Cor),

\therefore two \triangle^s are identically equal (Theor. 7) so that the $\angle DBO = \text{the } \angle OBE$.

That is, BO bisects the $\angle ABC$; or in other words, the centre O lies on the bisector of the $\angle ABC$.

7. Let AB and AC be two tangents to a circle whose centre is O . Join BC and AO cutting one another at D . It is reqd. to prove that AO bisects the chord of contact BC at rt. angles at D . Join OB, OC .



In the \triangle^s BOD and COD ,
because $\begin{cases} OB=OC \text{ (being radii)} \\ OD \text{ is common to both} \\ \text{and the } \angle BOD = \text{the } \angle DOC \text{ (Cor. Theor. 47),} \end{cases}$

\therefore two \triangle^s are identically equal (Theor. 4); so that the $\angle ODB = \text{the } \angle ODC$. The \angle^s ADB and ADC being adjacent angles, each is a rt. angle.

Hence AO bisects the chord BC at rt. \angle^s at D .

Q. E. D.

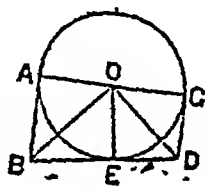
8. See figure in Ex. 4—Join PQ .

It is reqd. to prove that the $\angle PTQ = 2$ the $\angle OPQ$.

Proof—The $\angle OTQ = \text{the } \angle OPQ$ and $\angle OTP = \angle OQP$. (Theor. 39); also $\angle OPQ = \angle OQP$ (Theor 5); \therefore the $\angle PTQ = \text{the } \angle OTQ + \angle OTP = \angle OPQ + \angle OQP = 2\angle OPQ$.

Q. E. D.

9. Let two parallel CD touch the circle whose centre is O , at A and C . Let a third tangent, BD touching the circle at E , cut the parallel tangents AB and CD at B and D . Join OB and



tangents AB and AEC , whose centre is O . Let the third tangent, BD touching the circle at E , cut the parallel tangents AB , CD at B and D . Join OB and

It is reqd. to prove that the segment BD subtends a rt angle at the centre O , i.e., the $\angle BOD$ is a rt. \angle . Join OE .

Proof—Since BA and BE are tangents from B , $\therefore \angle AOB = \angle BOE$ (Theor. 47 Cor.), i.e., $\angle BOE = \frac{1}{2} \angle AOE$.

Similarly, it can be shown that $\angle EOD = \frac{1}{2} \angle EOC$.

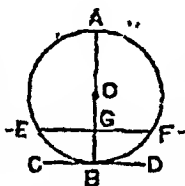
But the $\angle^s AOE$ and COE together $= 180^\circ$ (Theor. 14)

$$\therefore \angle^s BOE + EOD = \frac{1}{2} \times 180^\circ = 90^\circ$$

i.e., $\angle BOD$ is a rt angle. Hence BD subtends a rt. angle at the centre O .

Q. E. D.

10 Let AOB be the diameter of a circle whose centre is O , and let CD be the tangent to it at B .



It is reqd. to prove that the diameter AB bisects all chords parallel to the tangent CD .

Let EF be any chord parallel to CD cutting AB at G .

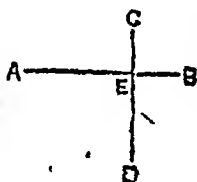
Proof—Since OB is perp to CD (Theor. 46), and EF is parallel to CD , $\therefore OB$ cuts EF at rt. angles (Ex. 3, page 41).

$\therefore OG$ bisects EF at G (Converse, Theor. 31).

Similarly, it can be proved, that AB bisects other chords parallel to CD .

Q. E. D.

11. Let AB be a given st. line. Let E be a given pt. in it. the locus of the centres of all circles which touch AB at E draw CE and DE at right angles to AB . Then CD is the



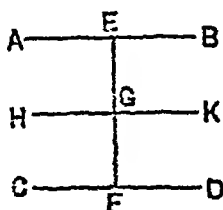
a given st. line and It is reqd. to find the locus of all circles which touch AB at the pt. E . Through E draw right angles to AB . Then CD is the reqd locus

Proof—Since the st. line CED is perp. to the tangent AB at the pt of contact E , it passes through the centres of the circles of which AB is a tangent at E (Cor. 2, Theor. 46).

Therefore CD is reqd. locus.

Q. E. F.

12. Let AB, CD be lines. It is reqd. the centres of all circles touching each of the st. lines $AB,$



any two parallel st. to find the locus of cles touching each CD . Take any pt.

F in the st. line CD .

At F draw FE perp. to CD meeting AB in E . Bisect EF at G .

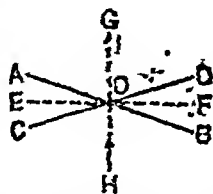
Through G draw HGK parallel to AB or CD . Then HK is the reqd. locus.

Proof—Since EF is perp. to CD , it is also perp. to AB (Ex. 3, page 41) Then a circle described with centre G and radius GE or GF will touch AB, CD at the pts. E, F respectively. (Theor. 46).

Thus it is evident that the centre of a circle touching two parallel st. lines is equi-distant from them; and HK is locus of each points. Hence HK is the reqd. locus.

Q. E. F.

13. Let two st. lines intersect to find the locus of circles which touch intersecting lines AB



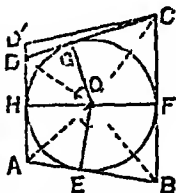
AB, CD of unlimi- at O . It is reqd. the centres of all each of the two in- and CD .

The centre of any circle which touches two intersecting st. lines lies on the bisector of the angle between them (Ex. 6).

\therefore the locus of the centres of all circles which touch each of two intersecting st. lines AB, CD is the pair of st. lines EF, GH which bisect the angles between the two given st. lines.

Q. E. D.

14. Let $ABCD$ be a quadrilateral circumscribed about whose centre is O , touch the circle at H .



be a quadrilateral the circle $EFGH$, and let the sides touch the circle at the pts. E, F, G and H .

It is reqd. to prove that $AB + DC = DA + CB$

Proof—Since from A two tangents AE, AH are drawn to the circle, $EFGH$, therefore $AE = AH$ (Theor. 47, Cor.)

Similarly $BE = BF, CG = CF$, and $DG = DH$.

$\therefore AE + BE + CG + DG = AH + BF + CF + DH$

or $(AE + BE) + (CG + DG) = (AH + DH) + (BF + CF)$

or $AB + DC = DA + CB$.

Q. E. D.,

Converse—If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, then a circle can be inscribed in it.

Let $ABCD$ be a quadrilateral in which $AB + DC = DA + CB$. It is reqd. to prove that a circle can be inscribed in $ABCD$. Bisect the $\angle^s DAB$ and ABC by st. lines AO, BO meeting at O .

Proof—Since AO, BO are the bisectors of the $\angle^s DAB$ and ABC , then O is the centre of the circle which would touch DA, AB and BC .

If this circle does not touch the side CD , let it touch the side CD' meeting AD , or AD produced at D' .

Then $AB + CD' = AD' + CB$ (proved)

But by hypothesis $AB + DC = DA + CB$

Subtracting the latter from the former, we have $CD' - DC = AD' - AD$ i. e., $CD' - DC = DD'$ or $CD' = DD' + DC$ which is absurd, (Theor. 11).

Hence the circle also touches the side CD ; therefore a circle can be inscribed in the quadrilateral $ABCD$.

Q. E. D.

15. See figure in Ex. 14.—Let $ABCD$ be a quadrilateral described about the circle $EFGH$ whose centre is O . Join OA , OB , OC and OD .

It is reqd. to prove that the \angle^s DOC and AOB subtended by DC and AB at $O = 2$ rt. angles; also the \angle^s DOA and COB subtended by AD and BC at $O = 2$ rt. angles. Join OE , OF , OH and OG .

Proof—Since the $\angle AOH =$ the $\angle AOE$ (Cor. Theor. 47), therefore the $\angle AOE = \frac{1}{2}$ the $\angle HOE$.

Similarly, the $\angle BOE = \frac{1}{2}$ the $\angle EOF$, the $\angle GOC = \frac{1}{2}$ the $\angle FOG$, and the $\angle DOG = \frac{1}{2}$ the $\angle GOH$.

$\therefore (\angle AOE + \angle BOE) + (\angle GOC + \angle DOG) = \frac{1}{2} (\angle HOE + \angle EOF + \angle FOG + \angle GOH)$

or $\angle AOB + \angle DOC = \frac{1}{2}$ of 4 rt. \angle^s (Cor. 2, Theor. 1),
 $= 2$ rt. \angle^s .

Similarly, it can be proved that the \angle^s $DOA + COB = 2$ rt. \angle^s .

Q. E. D.

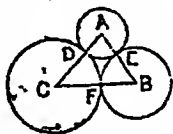
Page 179.

1. Take a st. line $AB = 2.6''$. With centres A and B , and radii $= 1.7''$ and $.9''$ respectively draw two circles. It will be found that the circles touch externally at a point D in AB , such that $AD = 1.7''$ and $DB = .9''$. They touch one another, because the sum of their radii $= 1.7'' + .9'' = 2.6'' =$ the distance between their centres [Theor. 48, Cor. (i)].

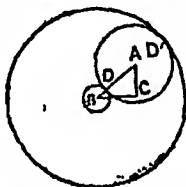
From AB cut off $AC = .8''$. With centre C and radius $= .9''$ draw a circle. It will be found that this

circle touches the circle, whose centre is A , internally at the pt. D . This circle touches the circle with centre A , because the difference of their radii, $1.7'' - .8'' =$ the distance between their centres [Theor. 48, Cor. (2)].

2. Construct the $\triangle ABC$ such that $BC=8$ cm., $AC=7$ cm. and $AB=6$ cm. (Prob. 8). With centres A , B , and C and radii $=2.5$ cm., 3.5 cm., and 4.5 cm. respectively draw three circles, touching in pairs at the pts. D , E and F , because $BC=8$ cm. $= (2.5 + 3.5)$ cm., $AC=7$ cm. $= (2.5 + 4.5)$ cm., and $AB=6$ cm. $= (3.5 + 4.5)$ cm. [Theor. 48, Cor. (2)].



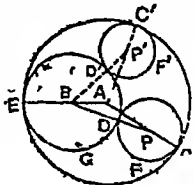
3. Take a st. line draw CA perp. to BC . Join AB . Then ABC angled triangle. With radius $=7$ cm. draw a st. D .



$BC=8$ cm. At C making $CA=6$ cm is the reqd. right-centre A and radius, cutting AB

Because $AB = \sqrt{BC^2 + AC^2} = \sqrt{8^2 + 6^2}$ or 10 cm., if a circle be drawn with centre B to touch the former circle internal and externally, then its radius will be $10 - 7 = 3$ cm. or $10 + 7 = 17$ cm. respectively.

4. Take a st. line $BA=2$ cm. With centres B and A , and radii $=3$ cm and 5 cm. respectively draw two circles EGD' and ECC' . Then these circles will touch each other internally at the pt E . Let P be the centre of the circle DFC which touches the circle EGD' externally at D and the circle ECC' internally at C . Join BE , AP , BD , DP and PC . Since A and P are the centres of the circles ECC' and DFC , and C is the pt. of contact of these two circles, therefore the pts A , P and C are in the same st. line (Theor. 48), *i.e.*, APC is a st. line. Again since B and P are the centres of the circles EGD' and DFC , and D is their pt. of contact BD and DP are in the same st. line (Theor. 48).

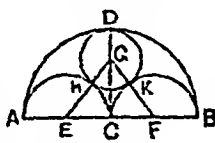


$AP = AC - PC$, $BP = BD + DP$, and $PC = DP$ being radii of same circle.

$\therefore AP + BP = AC + BD =$ sum of the radii of the given circles $=$ constant, and $= 5 + 3 = 8$ cm. in this case.

Similarly if P' be the centre of any other such circle, it can be proved that $AP' + BP' = 8$ cm.

5. Draw a st. line $AB = 4''$ and bisect it at C . Bisect AC at E and CB at F . With centres C , E and F and radii $= 2''$, $1''$ and $1''$ respectively describe the semi-circles ADB , AHC and CKB . Let G be the centre of the circle DHK touching the semi-circle ADB internally at D and the semi-circles AHC and CKB externally at the pts. H and K . Join DG , GC , GH , HE , GK and KF .



Since G and C are the centres of the circle DHK and the semi-circle ADB , and D is their pt. of contact, therefore the pts. D , G , C are in the same st. line (Theor. 48). i. e., DGC is a st. line. Similarly GH and HE , as well as GK and KF , are in the same st. line (Theor. 48). Since $AC = CB$, $AE = \frac{1}{2} AC$, and $CF = \frac{1}{2} CB$ therefore $EC = CF$, and hence $EH = KF$. $\therefore GH + HE = GK + KF$, or, $GE = GF$.

\therefore the $\triangle^s GEC$ and GCF are congruent (Theor. 7), so that the $\angle GCE =$ the $\angle GCF$; and these being adjacent angles each is a right angle.

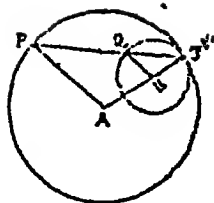
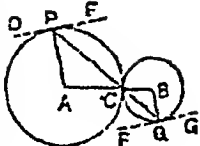
Let x'' be the length of the radius of the circle DHK , then $GC = DC - DG = (2 - x)''$, and $GF = GK + KF = (x + 1)''$.

Now, $GF^2 = GC^2 + CF^2$, or $(x + 1)^2 = (2 - x)^2 + 1^2$,

or, $x^2 + 2x + 1 = 4 - 4x + x^2 + 1$

or, $6x = 4$, $\therefore x = \frac{2}{3}$. $\therefore GD = \frac{2}{3}''$.

✓ 6 Let a st. line PCQ be drawn through C the pt of contact of two circles whose centres are A and B , cutting the circumferences at P and Q respectively. Join AP and BQ .



It is reqd to prove that AP and BQ are parallel. Join AC and CB

Proof— AC and CB are in the same st. line (Theor. 48).

Since $AP=AC$, and $BC=BQ$; therefore the $\angle APC =$ the $\angle ACP$, and the $\angle BCQ =$ the $\angle BQC$ (Theor. 5)

In the case when the two circles touch each other *externally*, the $\angle ACP =$ the $\angle BCQ$ (Theor. 3). Therefore the $\angle APC =$ the $\angle BQC$, and these being alternate angles, AP and BQ are parallel (Theor. 13).

In the case when the two circles touch each other *internally*, the $\angle ACP =$ the $\angle BCQ$, being the same angle. Therefore the int. $\angle APC =$ the ext. $\angle BQC$. Hence AP and BQ are parallel (Theor. 13).

Q. E. D.

7. See figure in Ex 6.—Let two circles whose centres are A and B touch externally at the pt. C , and through C , the point of contact let a st. line PCQ be drawn terminated by the circumferences. Let DPE and FQG be tangents to the circles at the pts P and Q respectively.

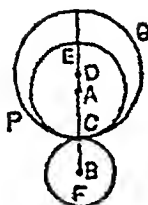
It is reqd. to prove that DE and FG are parallel. Join PA , AC , CB and BR .

Proof—Since AP and BQ are parallel (proved in Ex. 6), therefore the $\angle APQ =$ the $\angle PQB$ (Theor. 14). But the \angle s APE and BQF are equal, being rt. \angle s (Theor. 146).

\therefore the remaining $\angle QPE =$ the remaining $\angle PQF$, and these being alternate angles, DE and FG are parallel (Theor. 13)

Q. E. D.

8. (i) Let D be circle PCQ , and C a reqd. to find the locus of the centres of all the circles which touch at C .



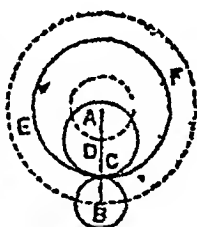
the centre of the given circle PCQ on it. It is the locus of the centres of all the circles which touch the given circle PCQ at C .

Let A be the centre of a circle touching the circle PCQ at C . Join DC , AC . Then D , A and C are in one st. line (Theor. 48).

i.e. A lies on CD or CD produced bothways, and since C and A are given pts. the line $EDCF$ is fixed. $\therefore A$ always lies on a fixed line EF , which is, therefore the reqd. locus.

Q. E. D.

(ii) Let A be the centre of the circle ECF , and let reqd. to find the locus of the centres of all the circles of a given radius touching the given circle ECF internally or externally. Let D and B be the centres of circles with the given radius touching the given circle ECF internally and externally at any pt C . Join AC , DC and BC .



centre of the given circle its radius be a . It is the locus of the centres of all the circles of a given radius (suppose b) and circle ECF , internal-

D and B be the centres of circles with the given radius touching the given circle ECF internally and externally at any pt C .

Since the circles with centres D and B touch the circle ECF internally and externally at C , therefore AC and DC , as well as AC and BC , are in one st. line (Theor. 48). Therefore AC , DC , and BC are in the same st. line.

Then $AD = AC - DC = a - b$, and $AB = AC + CB = a + b$. Now since a and b are constants therefore AD and AB are also constants; \therefore the distances of D and B from the fixed pt. A are always constants.

Hence the reqd. locus consists of the circles whose common centre is A , and radii equal to $(a - b)$ and $(a + b)$, as shown by dotted circles in the diagram.

The diagram shows a large circle representing a cell. Inside the cell, there are two smaller circles representing nuclei, labeled A and B. A horizontal line segment connects the centers of these two nuclei, with a point labeled D located between them. Another line segment, labeled EC, starts from a point E inside the cell and extends to a point C on the cell's boundary.

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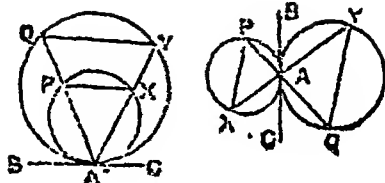
72°, then the $\angle BAD = 49^\circ = 72^\circ$. But the \angle s together $= 180^\circ$ (Theor. 40).
 $\angle BCD = 180^\circ - 72^\circ = 108^\circ$.
 $\angle BCD$ (Theor. 49) =

Proof.—The $\angle DAB =$ the $\angle ACB$ in the alt. segment, also, the $\angle DBA =$ the $\angle ACB$ (Theor. 49).

\therefore the $\angle DAB =$ the $\angle DBA$, and hence $DA = DB$ (Theor. 6).

Q. E. D.

3. Through A contact of two and APX let chords APQ be drawn terminating in circumference and QY.



the pt. of circles AQY any two and AXY terminated by es. Join BX

It is reqd. to prove that PX and QY are parallel. Draw BAC the common tangent to two circles at A.

Proof— i) For internal contact.

The $\angle BAP =$ the $\angle PXA$ in the alt. segment of the circle APX, and the $\angle BAQ =$ the $\angle QYA$ in the alt. segment of the circle AQY (Theor. 49).

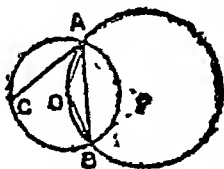
\therefore the ext. $\angle PXA =$ the int. $\angle QYA$, and hence PX and QY are parallel (Theor. 13).

(ii) For external contact.

The $\angle BAP =$ the $\angle PXA$ in the alt. segment of the circle APX, and the $\angle CAQ =$ the $\angle QYA$ in the alt. segment of the circle AQY (Theor. 49). But the $\angle BAP =$ the $\angle CAQ$ (Theor. 8), therefore the $\angle PXA =$ the $\angle QYA$ and these being alternate angles, PX and QY are parallel (Theor. 13).

Q. E. D.

4. Let A and intersection of which passes centre of the other; gent to the first P) at A. Join AB



B be the pts. of two circles one, of through O, the let CA be the tangent circle (with centre and OA.

It is reqd. to prove that OA bisects the $\angle CAB$. Join OB.

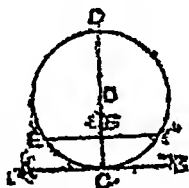
because $\begin{cases} \text{the } \angle ECG = \text{the } \angle ECF \text{ (proved)} \\ \text{the } \angle EGC = \text{the } \angle EFC, \text{ being rt. angles} \\ \text{and } EC \text{ is common to both} \end{cases}$

\therefore two \triangle^s are identically equal (Theor 17); so that $EG = EF$. Similarly it can be proved that $HF = HK$.

Q. E. D.

On the Method of Limits. Page 181.

2. Let DEC be a
diameter; let ACB
to DC at one of its



circle and DC its
be drawn perp.
extremity C.

It is reqd. to prove that AB is tangent to the circle at the pt. C. Draw any chord EF parallel to AB cutting DC at G.

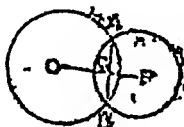
Proof—Since EF is parallel to AB, then DG is perp. to EF. Therefore EF is bisected at G (Converse, Theor. 31), and this is true *however closer G approaches to C.*

If the pt. G moves up to coincidence with C, then since $EG \text{ always} = GF$, the pts. E and F will coincide with the pt. C, and then the chord coincides with ACB, and cut the circle at one point only.

Hence, ultimately the st. line AB is a tangent at C.

Q. E. D

3. Let two circles
and P intersect at A



whose centres are O
and B Join AB

It is reqd. to prove that when the two circles touch one another the centres and the point of contact are in one st. line.

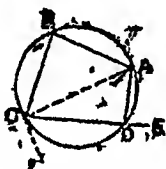
Proof— OP , the line of centres, bisects the common chord AB at right angles at C (given) *i.e.*, passes through C the mid pt. of AB . This is true *however near A and B approach to each other.*

If A and B come *very* close to one another and ultimately coincide, then since AC always $= CB$, the pt. C will also coincide with A and B , and the circles will touch each other at the pt. C .

Hence, ultimately the st. line which joins the centres of two circles touching each other, passes through the pt. of contact.

Q. E. D.

4. Let $ABCD$ be a cyclic quadrilateral, and let the side to any pt. E ; then $\angle ADE =$ the opp. \angle . $\angle ABC$ (Ex. 5, p. 163).



a cyclic quadrilateral CD be produced the ext. $\angle ADE = \angle ABC$ (Ex. 5, p.

It is reqd. to deduce Theorem 49, from the above data.

Proof—The $\angle ADE = \angle ABC$ (given). This is true *however near D approaches to C.*

If D moves up to and coincides with C , the chord AD will ultimately become the chord AC , the line CDE will become the tangent CE' , and the $\angle ADE$ will become the $\angle ACE'$.

Hence, ultimately the $\angle ACE' =$ the $\angle ABC$ in the alt. segment.

Q. E. D.

5. Let CAB be a circle. Take any pt. C on of the circle. Join AC is a \angle (Theor. 41) that the tangent at any perp. to the radius of contact,



and AB its diameter the circumference and BC . Then ACB It is reqd. to prove pt. of a circle is drawn to the pt.

Proof—the $\angle ACB$ is a rt. angle (given). This is true *however near C approaches to A*.

If C moves up to coincidence with A the chord BC will become the diameter BA, the chord CA will become the tangent AP, and the $\angle BCA$ will become the $\angle BAP$ $\therefore \angle BAP$ is a rt. angle.

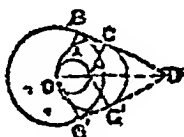
Hence the tangent PAQ at the pt. A of the circle AB C is perp. to the diameter BA (and therefore to the radius OA) drawn to the pt. of contact. Q. E. D.

Page 187.

1. There can be drawn—

(i) Two *direct common* tangents and no transverse, when the given circles intersect, (ii) Three common tangents—two *direct* and a third *at the point of contact*—when the circles have external contact, (iii) One *Common* tangent—at the point of contact—when the circles have internal contact.

(i) Draw a st. centres O and P, and $1''$ respectively. The circles intersect two points.

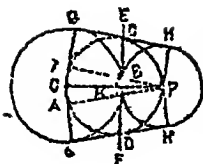


line $OP = 1''$. With and radii $= 1.4''$ draw two circles. They intersect one another at

Upon OP as diameter describe the circle AOA'P. With centre O and radius = the difference of two given radii ($1.4'' - 1'' = .4''$) draw arcs cutting the circle AOA'P at the pts. A and A'. Join OA and OA' and produce them to meet the circumference of the larger circle at B and B'. From P draw the radius PC parallel to OB and PC' parallel to OB'. Join BC and B'C'. Then BC and B'C' are the two *direct common* tangents.

There will be no transverse common tangents, for P will lie within the circle of construction for transverse tangents.

(ii) Draw a st. centres O and P , $1''$ respectively $GG'B$ and $H B H'$ each other exter- [Cor. (1) Theor.



line $OP = 2.4''$. With and radii $= 1.4''$ and draw two circles. The circles touch nally at the pt. B 48]

Upon OP as diameter describe the circle $CODP$. With centre O and radius $=$ the difference of two given radii ($1.4'' - 1'' = .4''$) draw arcs cutting the circle $CODP$ at A and A' . Join OA and OA' , and produce them to meet the circle $GG'B$ at G and G' . Draw the radii PH parallel to OG and PH' parallel to OG' . Join GH , $G'H'$. Then GH , $G'H'$ are the two direct common tangents. Draw EBF perp. to OP . Then EF is a transverse common tangent to the given circles at B , their pt. of contact, for P is on the circle of construction for transverse tangents.

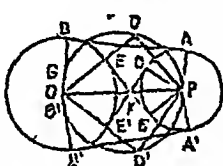
(iii) Draw a st. tres O and P , and respectively draw two each other inter-



line $PO = 4''$. With cen- radii $= 1.4''$ and $1''$ res- circles The circles touch nally at the pt. A [Cor.

(11) Theor. 48] Join PA . Then AP and PO are in one st. line (Theor. 48). Through A draw BAC perp. to AO . Then BC is the direct common tangent to the given circle at A their pt. of contact, since P is on the circle of construction. There are no transverse common tangents.

(iv) Draw a st. centres O and P , and $1''$ respectively $BE'EB$ and AA' neither cut nor



line $PO = 3''$. With and radii $= 1.4''$ draw two circles $C'C$. The circles touch each other.

Upon OP as diameter describe the circle $DPD'O$. With centre O and radius $=$ the difference of two given radii ($1.4'' - 1'' = .4''$) draw arcs cutting the circle $DPD'O$

Upon OP as diameter describe a circle. With centre P and radius = the difference of the given radii ($1\frac{7}{8}'' - 1''$) = $\frac{7}{8}''$ draw arcs cutting the circle (with diameter OP) at H and H' . Join PH , PH' and produce them to meet the circle $CEFC'$ at C and C' . Draw the radii OB parallel to PC and OB' parallel to PC' .

Join BC , $B'C'$. Then BC , $B'C'$ are the two *direct* common tangents.

$BC = O'H = \sqrt{OP^2 - HP^2} = \sqrt{2.1^2 - .7^2} = \sqrt{3.92} = 1.98''$ nearly. Also $B'C' = 1.98''$ nearly. Measure BC , $B'C'$, and it will be found that each of them = $1.98''$.

Join EF cutting OP at A . Let $OA = x$, then $AP = OP - OA = 2\frac{1}{2} - x$. But $OE^2 - OA^2 = AE^2 = EP^2 - AP^2$, or $1^2 - x^2 = 1.7^2 - (2.1 - x)^2$, or $3.2x = 2.52$,

$$\therefore x = .8''.$$

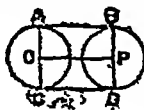
But $EF = 2OA = 2 \times .8 = 1.6''$. Measure EF and it will be found = $1.6''$.

Produce EF bothways to meet BC , $B'C'$ at D and D' respectively. Measure BD , DC , $B'D'$ and $D'C'$, and it will be found that $BD = DC$ and $B'D' = D'C'$. Hence DD' bisects the common tangents BC , $B'C'$.

5. See figure in Ex. 1, (iv)—Draw a st. line $OP = 3''$. With centres O and P , and radii = $1.6''$ and $.8''$ draw two circles. The circles neither cut nor touch each other.

Draw all the common tangents, as in Ex. 1, (iv). In this case OG or $OG' = (1.6'' - .8'') = .8''$ and OD or $OD' = (1.6'' + .8'') = 2.4''$.

6. Take a st. line OP of any length. With centres O and P and radii of equal length draw two equal circles. Through O and P draw AOC , BPD , diameters, of these circles, each perp. to OP . Join AB , CD . Then AB , CD are the two reqd. *direct* common tangents.



7. See figure in Ex. 1, (v). It is reqd. to prove that the two direct common tangents BC , $B'C'$ are equal. Join AP , $A'P$.

Proof—Since $OA = OA'$, OP is common, and $\angle^s = \angle OAP$ and $\angle OA'P$ are rt. angles, the two $\triangle^s OAP$ and $OA'P$ are equal (Theor. 18): so that $AP = A'P$. But $AP = BC$, and $A'P = B'C'$ $\therefore BC = B'C'$.

See figure in Ex. 1, (vi)—It is reqd. to prove that the two transverse common tangents CE and $C'E'$ are equal. Join PD , PD' .

Proof—Since the $\angle PDO$ is a rt. angle (Theor. 41) and the $\angle C'E'O$ is a rt. angle (Theor. 46), therefore the $\angle C'E'O =$ the $\angle PDO$. Hence PD , $C'E'$ are parallel (Theor. 13). But PC' is parallel to OD (by construction); therefore the figure $DPC'E'$, is a parallelogram. Therefore $PD = C'E'$ (Theor. 21). Similarly, $PD' = CE$.

But $PD = \sqrt{PO^2 - DO^2}$, $PD' = \sqrt{PO^2 - D'O^2}$ and $DO = D'O$ (by construction). $\therefore PD = PD'$, $\therefore C'E' = CE$.

Q. E. D.

8. See figure in Ex. 1, (i). Produce BC , $B'C'$ to meet at D . Join OD , PD . It is required to prove that OD and PD are in the same st. line.

Proof—The $\triangle^s BOD$ and $B'OD$ are identically equal (Theor. 18), because $OB = OB'$, OD is common to both and the $\angle OBD =$ the $\angle OB'D$ being rt. angles (Theor. 46), so that the $\angle BDO =$ the $\angle B'DO$. That is, OD bisects the $\angle BDB'$.

Similarly it can be proved that PD bisects the same angle. Therefore OD and PD are in the same st. line.

See figure in Ex. 1, (vi).—Let CE , $C'E'$ intersect at K . Join PK and KO . It is reqd. to prove that PK and KO are in the same st. line.

Proof—The $\triangle^s PCK$ and $PC'K$ are identically equal (Theor. 18), because $PC = PC'$, PK is common to both, and the $\angle PCK =$ the $\angle PC'K$ being rt. angles (Theor. 46); so that the $\angle PKC =$ the $\angle PKC' = \frac{1}{2} \angle CKC'$. Similarly it can be proved that the $\angle EKO =$ the $\angle E'KO = \frac{1}{2} \angle EKE'$. But the $\angle CKC' =$ the $\angle EKE'$.

Therefore the \angle^s PKC, PKC', EK'O and E'K'O are all equal

Now the \angle^s PKC' + PKC + CKE' = 2 rt. \angle^s (Theor. 1)

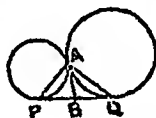
\therefore the \angle^s PKC + CKE' + E'K'O = 2 rt. \angle^s . Hence PK and K'O are in the same st. line (Theor. 2)

Q. E. D.

9. Let two given circles have external contact at A, and let PQ be a direct common tangent drawn to touch the circles at P and Q. Join AP, AQ. It is reqd. to prove that the \angle PAQ

is a rt. angle.

Let the common tangent to the two circles at A meet PQ



in B

Proof—Since BA and BP are two tangents from B, therefore $BA = PB$ (Theor. 47, Cor.). Therefore the \angle BAP = the \angle BPA. Similarly $BA = BQ$, therefore the \angle BAQ = the \angle BQA.

$\therefore \angle$ BAP + \angle BAQ = \angle BPA + \angle BQA, or \angle PAQ = \angle BPA + \angle BQA

\therefore the \angle PAQ is a rt. angle (Inference 4, Theor. 16)

Q. E. D.

On Loci. Foot of Page 188.

(i) See figure in Ex. 4, page 147.

The locus of the centres of the circles which pass through two given points is a straight line bisecting the line joining the two given points at right angles.

(ii) See figure in Ex. 11, page 177.

The locus of the centres of circles which touch a given straight line at a given point is a straight line perpendicular to the given straight line at the given point.

(iii) See figure in Ex. 8, (i), page 179.

The locus of the centres of circles which touch a given circle at a given point is the straight line passing through the centre of the given circle and the given point.

(iv) See figure in Ex. 12, page 177.

The locus of the centres of circles which touch a given straight line and have a given radius is the two straight lines parallel to the given straight line on either side of it and at a distance equal to the given radius from it.

(v) See figure in Ex. 8, (vi), page 179.

The locus of the centres of circles which touch a given circle and have a given radius is one or other of the two concentric circles whose radii are equal to the sum and difference of the two radii respectively.

(vi) See figures in Exs. 12 and 13, page 177.

The locus of the centres of circles which touch two given straight lines is a pair of straight lines bisecting the angles between the two given straight lines.

If the given straight lines are parallel, the locus is the straight line parallel to the given straight lines and midway between them.

Page 189.

1. Let A, B, C be any three given pts.
It is reqd. to draw a circle to pass through
A, B, C. Join AB, BC.

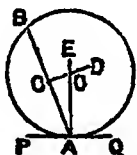


The centre of a circle passing through the pts. A, B lies on the st. line GD bisecting AB at rt. \angle^s [Note (i), page 188].

The centre of a circle passing through the pts. B, C lies on the st. line FE bisecting BC at rt. \angle^s [Note (1), page 188].

\therefore the pt. O where the st. lines GD, FE intersect satisfies both the conditions and is therefore the reqd. centre. With centre O and radius OA draw the circle which will also pass through B and C .

2. Let A be any pt. on the st. line PQ and B any other pt. outside it. It is reqd. to draw a circle to touch PQ at A and pass through the given pt. B . Join BA .

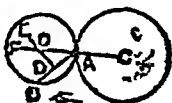


If a circle touches the st. line PQ at A , its centre lies on the st. line EA perp. to PQ at A [Note (11), page 188.]

If a circle passes through two given pts. A and B , its centre lies on the st. line DC bisecting AB at rt. \angle^s [Note (1), page 188].

\therefore the pt. O where the st. lines EA, DC intersect satisfies both the conditions, and is therefore the reqd. centre. With centre O and radius OA draw the circle which will touch the st. line PQ at A and pass through B .

3. Let C be the centre of the given circle and A any pt. on it. Let B be any other pt. outside the circle. It is reqd. to draw a circle to touch this circle at A and to pass through B . Join CA and produce it to any pt. F . Join AB .

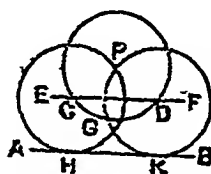


If a circle touches the given circle with centre C at the pt. A , its centre lies on the st. line through CA [Note (111), page 188].

If a circle passes through B, A its centre lies on the st. line ED bisecting BA at rt. \angle^s [Note (2), page 188].

\therefore the pt. O where the st. lines CF and ED intersect satisfies both the conditions, and is therefore the reqd. centre. With centre O and radius OA draw the circle which will touch the circle with centre C at A and pass through B.

4. Let P be a pt. 4.5 cm. from a given st. line AB is reqd. to draw two circles of radius 3.2 cm. to pass through P and touch AB



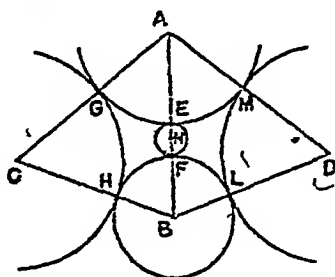
at a distance of st. line AB. It circles of radius through, P and

Locus of the centres of circles of radius 3.2 cm. which touch the given st. line AB is a st. line EF parall^l to AB situated at a distance of 3.2 cm. from it [Note (iv), page 188].

Locus of the centres of circles of radius 3.2 cm. which pass through P is a circle CGD, with centre P and radius = 3.2 cm. Let these two loci intersect at C and D.

Then C and D satisfy both the conditions, and are therefore the reqd. centres. With centres C, D and radius = 3.2 cm. draw two circles which will pass through P and touch AB at H, K.

5. Draw a 6 cm. With and radii = 3 respectively. It is reqd. circle of radius 3.5 each of the externally.



st. line AB = centres A, B, cm. and 2 cm. draw two circles to draw a circle to touch given circles

Locus of the centre of a circle of radius 3.5 cm. touching the given circle of radius 3 cm. externally is a circle whose centre is A and radius = $(3 + 3.5) = 6.5$ cm. [Note (v), page 188].

Locus of the centre of a circle of radius 3.5 cm. touching the given circle of radius 2 cm. externally is a circle whose centre is B and radius $= (2 + 3.5) = 5.5$ cm. [Note (v), page 188].

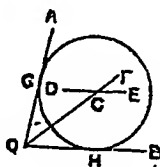
With centres A and B, and radii $= 6.5$ cm. and 5.5 cm. respectively draw arcs on either side of AB cutting at C and D.

Then C and D are the reqd. centres. With centres C and D, and radius $= 3.5$ cm. draw two circles. These circles will touch the two given circles at G, H, M, and L. Thus there are two solutions of this problem.

The centre N of the smallest circle, which touches the given circles with centres A and B externally, lies on AB midway between the pts. E and F where the given circles cut AB.

$EF = AB - AE - FB = 6 - 3 - 2 = 1$ cm. Therefore EN the radius of the smallest circle $= \frac{1}{2}$. EF = 5 cm.

6. Make the \angle reqd. to describe a circle to touch the lines OA,



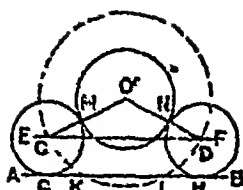
$\angle AOB = 76^\circ$. It is a circle of radius 1.2" touching OB.

If a circle touches two st. lines OA, OB its centre lies on OF, the bisector of the $\angle AOB$ [Note (vi), page 188].

Locus of the centre of a circle of radius 1.2" and touching the st. line OB is a st. line DE parallel to OB at a distance of 1.2" from it. [Note (v), page 188].

\therefore the pt. C where the st. lines FO, DE intersect is the reqd. centre. With centre C and radius $= 1.2''$ draw a circle which will touch OA, OB at G, H respectively.

7. Let O be the centre of a given circle of radius 5 cm., and AB a st. line. It is reqd. to draw two circles of radius 2.5 cm. touching the given circle and the given st. line AB .

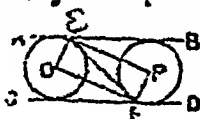


centre of a circle of radius 3.5 cm. at a distance of 5 cm. from a given st. line. It is reqd. to draw two circles of radius 2.5 cm. touching the given circle and the given st. line AB .

Locus of centres of circles of radius 2.5 cm. touching the given st. line AB is a st. line EF parallel to AB , at a distance of 2.5 cm. from it. [Note (iv), page 188].

Locus of centres of circles of radius 2.5 cm. touching the given circle with centre O is one or other of two circles whose common centre is O and radius = $(3.5 + 2.5)$ or 6 cm. and $(3.5 - 2.5)$ or 1 cm. respectively. [Note (v), page 188] The first circle $KCDL$ cuts EF at C and D ; but the other does not. Then C and D are the reqd. centres. With centres C, D and radius = 2.5 cm. draw two circles which will touch the given circle at M, N and the given st. line AB at G, H .

8. Let AB, CD be any two parallel st. lines and EF any transversal cutting AB, CD at E, F respectively. It is reqd. to draw a circle to touch AB, CD and EF .



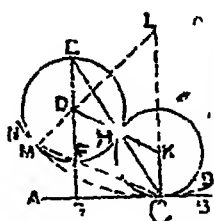
Locus of the centres of circles touching EF and CD is one or other of the st. lines FO, FP bisecting the \angle^s EFC, EFD respectively [Note (vi), page 188].

Locus of the centres of circles touching FE and AB is one or other of the st. lines EO, EP bisecting the \angle^s AEF, FEB respectively [Note (vi), page 188].

Hence O where FO, EO meet, and P where FP, EP meet are the reqd. centres. Since the circles touch AB and CD , their centres are equi-distant from AB and CD . Hence their radii are each = half the perp. distance between AB and CD . Draw the reqd. circles.

These circles are equal, because their radii are equal.

9. Let EMH whose centre is D , in a given st line to draw a circle to circle EMH , and C .



be a given circle and C a given pt. AB It is reqd. to touch the given st line AB at

Construction—At C draw CK perp. AB then the centre of the reqd. circle lies on CK [Note (22), page 188]. From D the centre of the circle EMH draw DG perp. to AB cutting the circle at F . Produce GD to meet the circle again at E . Join EC cutting the circle at H . Join DH and produce it to meet CK at K . Then K is the centre of the reqd. circle.

Proof—Since EG, KC are both \perp to AB , therefore they are parallel (Ex. 2, page 41)

\therefore the $\angle DEH =$ the alt. $\angle HCK$ (Theor. 14). Again since $DE = DH$, therefore the $\angle DEH =$ the $\angle DHE$. (Theor. 5)

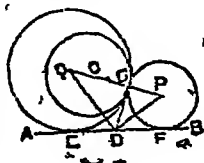
\therefore the $\angle HCK =$ the $\angle DHE =$ the vertically opp. $\angle KHC$.

$\therefore KH = KC$. Draw a circle with centre K and radius KH . Then this circle will touch the given circle EMH at H and the given st. line AB at the given pt. C

Another circle can be drawn to satisfy the given conditions. Join CF and produce it to meet the circle at M . Join DM . Produce MD, CK to meet at L . Then L is the centre of the reqd. circle

Since $DM = DF$, the $\angle DMF =$ the $\angle DFM$ (Theor. 5) = the vertically opp. $\angle GFC$ (Theor. 3) = the alt. $\angle FCK$. Therefore $LM = LC$

10. Let AB and C a given circle whose centre to draw a circle also the given OC . At C draw



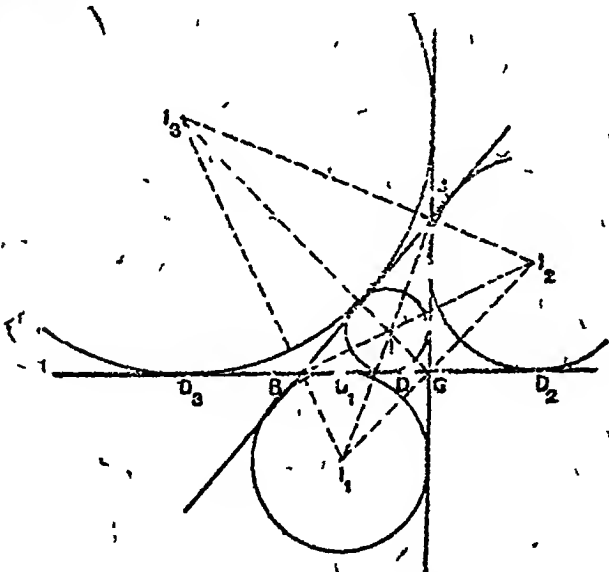
be a given st line pt. on a given cir. is O It is reqd. to touch AB , and circle a C . Join CD perp. to OC ,

meeting AB in D . Then CD will be the common tangent to the given circle and the reqd. one.

Centre of the reqd circle touching the given circle at C lies on the st. line through O, C . [Note (ii), page 188]

Again the centre of the reqd. circle which touches the st. lines CD, AB lies on one or other of the st. lines DP, DQ the bisectors of the $\angle^s CDB, CDA$ respectively [Note (vi), page 188]

\therefore the pts. P, Q where OC produced bothways meet DP, DQ are the reqd. centres. With centres P, Q and radii PC, QC respectively, draw two circles which touch the given circle at C and the given st. line AB at F, E .



11. Let AB, BC, CA be three given st. lines of which no two are parallel. It is reqd. to draw circles to touch each of these given st. lines.

(1) Locus of centres of circles touching the st. lines AB and BC is one or other of the st. lines $BI_2, I_1 BI_3$, the bisectors of the angles between AB and BC [Note (vi) page 188].

(2) Locus of centres of circles touching the st. lines BC, CA is one or other of the st. lines $CI_3, I_2 CI_1$ the

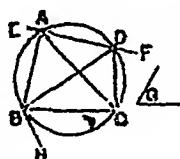
bisectors of the angles between BC and CA [Note (vi), page 188].

Let BI_3, CI_3 meet at I_3 ; CI_1, BI_1 at I_1 , BI_2, CI_2 at I_2 , and BI_2, CI_3 at I . \therefore the pts. I, I_1, I_2 and I_3 satisfy both the conditions; \therefore they are the centres of the reqd. circles. With centres I, I_1, I_2 and I_3 draw the circles as in diagram.

Thus there are four circles to touch each of the three given st. lines AB, BC, CA

Page 191.

1. Let BC be given angle and It is reqd to describe a triangle upon BC , having its vertex on the

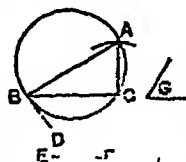


the given base, & the EF the given st. line. describe a triangle upon vertical angle $= \angle G$ line EF .

Upon BC describe a segment $BADC$ containing an angle $=$ the $\angle G$ (Prob. 24). Then the vertex of the reqd. triangle lies on the arc $ABDC$. Also the vertex lies on the st. line EF . Therefore, the pts A, D where the segment $BADC$ cuts the st. line EF represent the vertices of the reqd. triangle. Join AB, AC, DB, DC . Then ABC, DBC are the two reqd. triangles.

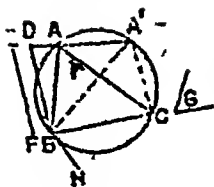
2. Let BC be the given base and G the given vertical angle. Upon BC describe a segment BAC containing an angle $=$ the given $\angle G$ (Prob. 24). Then the vertex of the triangle whose base is BC and the vertical angle $= \angle G$ lies on the arc BAC .

(1) Let EF describe one of the sides of centre B and radius BC . Then the vertex of lies on this arc. A where this BAC is the reqd. vertex. Join AB, AC . Then ABC is the reqd. triangle.



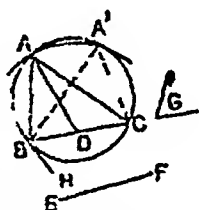
note the length of the triangle. With $= EF$ draw an arc. the reqd. triangle. Therefore the pt. arc cuts the arc

(i) Let EF denote altitude. At B draw making $BD = EF$. parallel to BC . Then reqd. triangle also DA' . Therefore the AD' cuts the arc BAC are the reqd. vertices. Join AB , AC , $A'B$, $A'C$. Then ABC , $A'BC$ are the reqd. triangles.



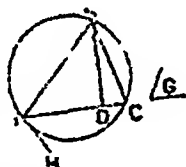
the length of the BD perp. to BC . From D draw DA' the vertex of the lies on the line pts. A , A' where

(ii) Let EF denote median which bisects at D . With centro EF draw an arc. Then the reqd. triangle also. Therefore the pts. A , arc cuts the arc BAC vices. Join AB , AC , $A'B$, $A'C$. Then ABC , $A'BC$ are the reqd. triangles.



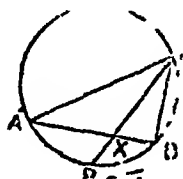
the length of the BC . Bisect BC D and radius = the vertex of lies on this arc. A' where this are the reqd. vertices.

(iv) Let D be the from the vertex to the draw DA perp. to BC . of the reqd. triangle line DA . Therefore DA meets the arc BAC is the reqd. vertex. Join BA , AC . Then ABC is the reqd. triangle.



foot of the perpen. base BC . At D Then the vertex lies on the st. the pt. A where

3. Construct the in the Text Book.



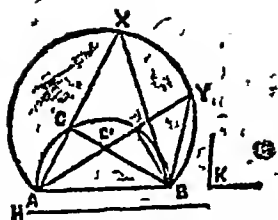
triangle as given

Proof—Since the arc AP = the arc PB (by construction), therefore the $\angle ACP$ = the $\angle PCB$ (Theor 43). Hence the st. line CP is the bisector of the vertical $\angle ACB$ which is equal to the given $\angle K$. Therefore ABC is the reqd. triangle.

4. Construct

given in the Text

BX.

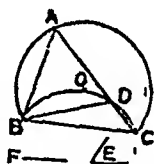


Book.

Join

Proof—Since the $\angle ACB = \text{the } \angle K$, the $\angle BXC = \frac{1}{2} \angle K$, and $\angle ACB = \angle BXC + \angle XBC$ (Theor. 16), therefore $\angle XBC = \angle ACB - \angle BXC = \angle K - \frac{1}{2} \angle K = \frac{1}{2} \angle K = \text{the } \angle BXC$. $\therefore CX = CB$ (Theor. 6). $BC + CA = AC + CX = AX = H$. Therefore ABC is the reqd. triangle. AY cuts the smaller segment at C' . Join AC' , BC' and BY . Then it can be proved that ABC' is another such triangle.

5. Let BC be the given base, E the given angle and the difference of the



given base, E the line F equal to remaining sides.

Construction—On BC describe a segment BAC containing an angle equal to E , also another segment BOC containing an angle $\angle 90^\circ + \frac{1}{2} E$, (Prob 24). With centre C and radius $= F$ draw an arc cutting the arc BOC at D . Join CD and produce it to meet the arc BAC at A . Join AB . Then ABC is the reqd. triangle.

Proof—The $\angle ADB = 180^\circ - \angle BDC = 180 - (90^\circ + \frac{1}{2} E) = 90^\circ - \frac{1}{2} E$

The $\angle BDC = \angle BAD + \angle ABD$, (Theor. 16), therefore the $\angle BAD = \angle BDC - \angle ABD = (90^\circ + \frac{1}{2} E) - E = 90^\circ - \frac{1}{2} E$. Therefore the $\angle ADB = \text{the } \angle BAD$, and hence $AD = AB$ (Theor 6). $AC - AB = AC - AD = DC = F$. $\therefore ABC$ is the reqd. Δ .

The $\angle GAB = \angle ACB$ in the alt. segment (Theor. 49) $= 60^\circ$; and the $\angle HAC = \angle ABC$ in the alt. segment (Theor. 49) $= 60^\circ$. But the $\angle^s GAB, BAC$ and $CAH = 180^\circ$ (Theor. 1). Therefore the $\angle BAC$ also $= 60^\circ$. Hence the $\triangle ABC$ is equiangular, and consequently equilateral (Cor. Theor. 6).

Since the \angle^s POQ, QOR, POR = 360° (Cor. 3, Theor. 1) therefore the \angle POR = $360^\circ - 240^\circ = 120^\circ$. Since the \angle^s OPD, ORD are rt. angles (Theor. 46), therefore the pts. D, P, O, R are concyclic (Converse, Theor. 40): Therefore the \angle^s PDR, POR = 180° (Theor. 40). Therefore the \angle PDR = $180^\circ - 120^\circ = 60^\circ$. Similarly it can be shown that the \angle^s PEQ, RFQ each = 60° . Hence the $\triangle DEF$ is equiangular, and consequently equilateral (cor. Theor. 6).

BC = 8 cm. With radius = 8 cm. draw another at A. Join is the reqd. equilate-

Bisect the \angle^s BAC, ACB by the st. lines AK, CH cutting one another at O. Then O is the centre of the

inscribed circle (Prob. 26). Let AK , CH meet BC , AB at K and H .

$\therefore AK$ bisects BC at rt. angles, and CH bisects AB at rt. angles (Ex. 1 page 19).

And AK and CH cut one another at O . Therefore O is also the centre of the circumscribed circle (Prob. 25)

Produce AB , AC to F and G . Bisect the $\angle^s CBF$, BCG by the st. lines BP , CP meeting each other at P . Then P is the centre of the escribed circle (Prob. 27).

Join PK . The $\angle FBC = 120^\circ = \angle BCG$. Therefore their halves are equal, so that $\angle PBC = \angle PCB = 60^\circ$, hence $PB = PC$ (Theor. 6). The $\triangle^s BKP$, PKC are identically equal (Theor. 7), because $PB = PC$, $BK = KC$ (proved) and PK is common to both, so that the $\angle BKP = \angle PKC$ and these being adjacent angles, each is a rt. \angle . But the $\angle BKA$ is also a rt. \angle . Therefore AK and PK are in one st. line.

The $\triangle^s BAK$ and BKP are congruent, because $\angle BKA = \angle BKP$ being rt. angles, $\angle ABK = \angle KBP$ being $= 60^\circ$, and BK is common to both (Theor. 17) Therefore $AK = KP$. Since AK is the median of the $\triangle ABC$, $OK = \frac{1}{3} AK$, $AO = \frac{2}{3} AK$ (Cor. Proposition III, page 97).

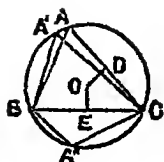
$\therefore AO = 2 OK$, and $KP = 3 OK$. Hence the circum-radius OA and ex-radius PK are respectively double and treble of the in-radius OK . $AK = \sqrt{AC^2 - KC^2} = \sqrt{8^2 - 4^2} = \sqrt{48} = 6.9$ cm. nearly.

$OK = \frac{1}{3} \times 6.9 = 2.3$ cm., $\therefore OA = 4.6$ cm. and $PK = 6.9$ cm.

Measure them and it will be found that $OK = 2.3$ cm., $OA = 4.6$ cm. and $PK = 6.9$ cm.

(4) Draw a st. line $BC = 2.5''$. At B , C make the $\angle^s CBA$, $BCA = 66^\circ$ and 50° respectively, the arms BA , CA meeting at A . Then $\triangle ABC$ is the reqd. triangle. Bisect

BC, AC at E and D.
EO, DO meeting
centre O and radius
which will pass
(Prob. 25). Measure
found to be $\approx 1.39''$



At E, D draw perps.
each other at O. With
OB draw a circle
through C and A also
OB and it will be

(ii) Draw the $\triangle A'BC$ making $\angle B=72^\circ$, $\angle C=44^\circ$.

(iii) Also draw the $\triangle A''BC$ making $\angle B=41^\circ$, $\angle C=23^\circ$ but on the other side of AC. Circumscribe a circle in each case and measure the radius which will be found to be $1.39''$ in each case. The vertical $\angle A=180^\circ - B+C$,
(Theor. 16, $=180^\circ - (66^\circ + 50^\circ) = 64^\circ$ in case (i).

$$\angle A' = 180^\circ - (72^\circ + 44^\circ) = 64^\circ \text{ in case (ii),}$$

$$\angle A'' = 180^\circ - (41^\circ + 23^\circ) = 116^\circ \text{ in case (iii).}$$

Because the base BC is of the same length in all the cases, and the vertical $\angle A$ in case (i) = the vertical $\angle A'$ in case (ii) = the supplement angle of the vertical $\angle A''$ in case (iii), therefore they lie on the same circle (Theors. 39 and 40, converses). Hence their circum-radii are equal.

4. See figure in Ex. 1.—With any pt. O as centre and radius = 4 cm. describe a circle. Inscribe and circumscribe equilateral $\triangle^s ABC, DEF$ in and about this circle, as in Ex 1. Draw AK perp. to BC. The $\triangle^s ABK, ACK$ are congruent, because $AB=AC$, AO is common to both, and the $\angle AKB = \angle AKC$ being rt. angles (Theor. 18), therefore $BK=KC$. That is, AK bisects the base BC. Join OK, then it is perp. to BC (Theor. 31). Therefore AK, OK are in one st. line. Join CO and produce it to meet AB at M. It can be proved that CM is a median of the $\triangle ABC$. $OK = \frac{1}{2} AO$, (Cor Prop. III, page 97) = 2 cm.

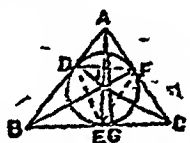
Hence $KC = \sqrt{OC^2 - OK^2} = \sqrt{4^2 - 2^2} = 3.46$ cm.
Therefore $BC = 2 \times 3.46 = 6.9$ cm. nearly. Measure BC and it will be found to be 6.9 cm. $AK = AO + OK = 4 + 2 = 6$ cm. Hence the area of the $\triangle ABC = \frac{1}{2} \cdot BC \cdot AK = \frac{1}{2} \times 6.9 \times 6 = 20.7$ sq. cm.

Since O is the in centre of the $\triangle DEF$, DO bisects the $\angle EDF$ (Prob. 26). \therefore DO when produced would bisect EF at rt. angles. (See Ex 1, page 19).

Again since both DO produced and OQ are perp. to EF from O, DO and OQ are in the same str. line; i. e., DOQ is a median of $\triangle DEF$. Similarly it can be shown that FOP is also a median of $\triangle DEF$. $\therefore DQ = 3 OQ = 12$ cm. Also $FP = 3 OP = 12$ cm. $\therefore FO = \frac{2}{3} FP = 8$ cm. $\therefore QF = \sqrt{FO^2 - OQ^2} = \sqrt{8^2 - 4^2} = 6.9$ cm

\therefore area of the $\triangle DEF = \frac{1}{2} EF \cdot DQ = \frac{1}{2} \times 12 \times 6.9$ or 82.8 sq. cm. $= 4 \times 20.7$ sq. cm. $= 4 \triangle ABC$.

5. Let ABC be a $\triangle ABC$, $\angle A$ bisected by the meeting at I. Then the inscribed circle ID, IE, IF perp. on reepectively. Since ID, IE, IF are radii of the inscribed circle, therefore each of them $= r$. Join IA.



triangle. Bisect st. lines BI, CI. I is the centre of (Prob. 26). Draw AB, BC and CA

$\triangle IAB = \frac{1}{2} ID \cdot AB = \frac{1}{2} cr$, $\triangle IBC = \frac{1}{2} IE \cdot BC = \frac{1}{2} ar$, and $\triangle ICA = \frac{1}{2} IF \cdot AC = \frac{1}{2} br$.

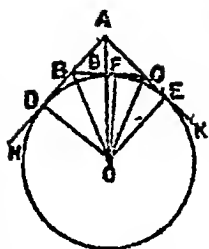
But the $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$
 $= \frac{1}{2} ar + br + cr = \frac{1}{2} (a + b + c) r$.

In the $\triangle ABC$, if $AB = 9$ cm., $BC = 8$ cm., and $AC = 7$ cm., then ID will be found to be 2.24 cm. on measuring.

$\therefore \triangle ABC = \frac{1}{2} (a + b + c) r = \frac{1}{2} (9 + 8 + 7) \times 2.24 = 26.8$ sq. cm. Draw AG perp. to BC. Then AG will be found to be 6.7 cm. (see page 111). In this case the $\triangle ABC = \frac{1}{2} AG \cdot BC = \frac{1}{2} \times 6.7 \times 8 = 26.8$ sq. cm.

Thus it is evident that the formula— $\triangle ABC = \frac{1}{2} (a + b + c) r$ —is true.

6. Let $\triangle ABC$ be the sides AB, AC to bisect the $\angle^s CBH, BO, CO$ meeting at centre of the escribed circle opposite to A . From perps. to AH, AK , Since OD, OE, OF are the radii of the escribed circle, therefore each of them $= r_1$. Join AO .



triangle. Produce any pts. H and K BCK by the st. lines O . Then O is the ed circle (Prob. 27) O draw OD, OE, OF BC respectively. are the radii of the

$\triangle ABO = \frac{1}{2} OD \cdot BA = \frac{1}{2} cr_1$, $\triangle ACO = \frac{1}{2} OE \cdot AC = \frac{1}{2} br_1$, and $\triangle BCO = \frac{1}{2} OF \cdot BC = \frac{1}{2} ar_1$.

But the $\triangle ABC = (\triangle ACO + \triangle ABO) - \triangle BCO = (\frac{1}{2} br_1 + \frac{1}{2} cr_1) - \frac{1}{2} ar_1 = \frac{1}{2} (b + c - a) r_1$

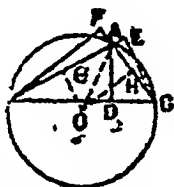
In the $\triangle ABC$, if $BC = 5$ cm., $AC = 4$ cm., and $AB = 3$ cm., then OF will be found to be 6 cm. on measurement.

$\therefore \triangle ABC = \frac{1}{2} (b + c - a) r_1 = \frac{1}{2} (4 + 3 - 5) \times 6 = 6$ sq. cm.

Draw AG perp. to BC , then it will be found to be 2.4 cm. (see page 111 of the Book). In this case the $\triangle ABC = \frac{1}{2} AG \cdot BC = \frac{1}{2} \times 2.4 \times 5 = 6$ sq. cm.

Thus it is evident that the formula $\triangle ABC = \frac{1}{2} (b + c - a) r_1$, is true.

7. Construct the 6.3 cm., $b = 3$ cm., (Prob. 8). Bisect at G, H respectively. HO perps. to AB ,



$\triangle ABC$ in which $a =$ and $c = 5.1$ cm. the sides AB, AC At G, H draw GO, AC meeting each

other at O Then O is the centre of the circle circumscribed about the $\triangle ABC$ (Prob. 25). Join OA and measure it; it will be found to be 3.2 cm. nearly.

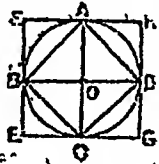
From A, B, C draw AD, BF, CE perps to BC, AC, AB respectively. Measure AD, BF and CE , and it will be found that $AD = 2.4$ cm., $BF = 5.04$ cm., and $CE = 2.96$ cm.

If AD , BF , CE be represented by p_1 , p_2 and p_3 respectively, then $\frac{bc}{2p_1} = \frac{3 \times 5.1}{2 \times 2.4} = 3.2$ cm. nearly, $\frac{ca}{2p_2} = \frac{5.1 \times 6.3}{2 \times 5.04} = 3.2$ cm. nearly and $\frac{ab}{2p_3} = \frac{6.3 \times 3}{2 \times 2.96} = 3.2$ cm. nearly.

\therefore The circum-radius $OA = 3.2$ cm. $= \frac{bc}{2p_1} = \frac{ca}{2p_2} = \frac{ab}{2p_3}$.

Page 199.

1. With centre O describe a circle, and in meters AC , BD at rt. Join AB , BC , CD , the reqd. square, since (Theor. 41), and any



and radius $= 1.5''$ describe it take any two diameters to each other. DA . Then $ABCD$ is its \angle^s are all rt. \angle^s side, say $BC =$

$$\sqrt{BO^2 + OC^2} = \sqrt{2OB^2} = OB\sqrt{2} = 1.5\sqrt{2}, \text{ or } 2.12''.$$

Measure BC and it will be found to be $2.12''$ long.

Area of the square $= BC^2 = 2OB^2 = 2 \times (1.5)^2$, or 4.5 sq. in.

2. See fig in Ex 1.—With centre O and radius $= 1.5''$, draw a circle, and take in it two diameters AC , BD at rt. angles to each other. Draw tangents at the pts. A, B, C, D , cutting one another at E, F, G, H . Then $EFGH$ is the reqd. circumscribed square. Join AB , BC , CD , DA .

Since EH , BD and FG are at rt. angles to the same st. line AB , \therefore they are parallel. Similarly EF , AC and HC are parallel.

\therefore each of the figs. $EFGH$, $EHDB$, $BDGF$, $AEFC$ is a Parallelogram.

$$\therefore FG = EH = BD = AC = EF = GH.$$

Now $\angle EBD$ is a rt. \angle , \therefore the Parallelogram $EHDB$ is a rectangle.

$\therefore \angle BEH$ is also a rt. \angle . $\therefore EFGH$ is a square.

Because the rectangles $EBDH$ and $BDGF$ are respectively double of the $\triangle^s ABD$ and BDC ,

\therefore the whole square $EFGH =$ twice the square $ABCD$

3. See fig in Ex 1.—take a line $EF = 7.5$ cm., and on it describe a square (Prob. 13.). Bisect EF, FG at B, C , at B, C draw BD, CA perps. to EF, FG intersecting at O . With centre O and radius $= OB$ describe a circle, it will touch the sides at A, B, C and D .

In the fig. $BOCF$ since the $\angle^s BFC, OBC, OCF$ are rt. \angle^s , $\therefore BOCF$ is a rectangle, $\therefore OB = OC$, also $\therefore \angle BOC =$ a rt. \angle . \therefore each of the angles at O are rt. angles.

Fold the square about AC ; then since $\angle^s ACF$ and ACG are rt. \angle^s , CF will fall on CG ; and because $CF = CG$, F will fall on G . Now since $\angle^s CFB$ and CGD are equal (being rt angles), FB falls on GD

Again since $\angle^s BOC$ and COD are rt. \angle^s , OB falls on OD .

$\therefore B$ falls on D $\therefore OB = OD$; and $\angle ODG = \angle OBF =$ a rt. \angle . Hence the circle touches GH at D .

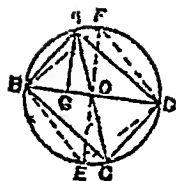
Similarly, it can be proved that $OA = OC (= OB, \text{ proved})$ and $\angle OAE = \angle OCF =$ a rt. \angle . Hence the circle touches FG and EH at C and A . \therefore it is the inscribed circle.

4 See Fig. in Ex. 1.—Draw a square $ABCD$ on a line $AB = 6$ cm. Join AC, BD cutting one another at O . Then the diagonals AC and BD are equal, and they bisect one another at O . $\therefore OA = OB = OC = OD$. With centre O and radius $= OA$ describe a circle; then it will pass through B, C, D also. \therefore it is the circumscribed circle.

Measure the diameter BD , and it will be found to be 8.5 cm. long.

By calculation, $BD = \sqrt{AB^2 + AD^2} = AB\sqrt{2} = 6\sqrt{2}$, or 8.48 cm.

5. With any pt. O
 $=1.8''$ draw a circle.
 center on the cir-
 cums= $3''$ draw an arc
 D Join AD , and at
 lines AB, DC perp.
 circle at B and C .



CD is the reqd. rectangle. Join AC, BD ; they are diagonals since $\angle^s ADC, BAD$ are rt. \angle^s then they cut one another at centre O .

The side $DC = \sqrt{AC^2 - AD^2} = \sqrt{3.6^2 - 3^2} = 1.99''$ or $2''$ nearly.

Draw the diameter FOE perp. to BD . Join FB, FD BE and ED . Then $FBED$ is a square inscribed in the circle. Draw AG perp. to BD .

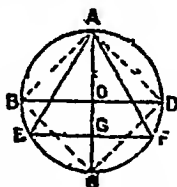
Area of the sq. $FBED = 2$ the $\triangle FBD = FO \cdot BD$; and area of the rect. $ABCD = 2$ the $\triangle ABD = AG \cdot BD$.

Now, AO being the hypotenuse is greater than AG . But $FO = AO$ (being radii) $\therefore FO$ is greater than AG . $\therefore FO \cdot BD$ is greater than $AG \cdot BD$.

Therefore the area of the sq. $FBED$ is greater than the area of the rect. $ABCD$.

Likewise it can be proved that the sq. $FBED$ is greater than any other inscribed rectangle. Hence of all the rectangles inscribed in a circle the square has the greatest area.

6. Let $ABCD$ be a
 equilateral triangle
 given circle, and let
 lengths of their sides.



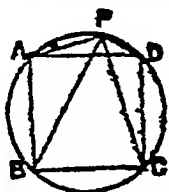
square and AEF an
 inscribed in the
 a and b denote the

If r denote the radius of the given circle, then $BD^2 = AB^2 + AD^2$, or $(2r)^2 = a^2 + a^2 = 2a^2$, or $r^2 = \frac{1}{2} a^2$.

$$AE^2 = AG^2 + EG^2 \text{ or } b^2 = (r + \frac{1}{2} r)^2 + (\frac{b}{2})^2, \text{ or } \frac{3}{4} b^2 = \frac{3}{4} r^2; \therefore r^2 = \frac{1}{3} b^2 \{ EG = AO + OG = AO + \frac{1}{2} AO = r + \frac{1}{2} r \}$$

$$\therefore \frac{1}{3} a^2 = \frac{1}{3} b^2, \therefore 3 a^2 = 2 b^2.$$

7 Let ABCD be a circle and let P AD Join PA, PD, to prove that the times any one of the CPD.

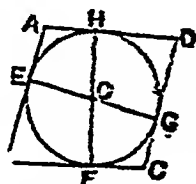


a square inscribed in be any pt. on the arc PB, PC. It is reqd $\angle APD = \text{three } \angle^s \text{ APB, BPC and CPD.}$

Proof—Since the chords AB, BC, CD are equal to one another, the arcs AB, BC, CD are also equal (Theor. 41); and hence the \angle^s APB, BPC, CPD subtended by these equal arcs are equal (Theor. 43).

Hence the $\angle APD = \angle APB + \angle BPC + \angle CPD = 3$ times any one of the \angle^s APB, BPC, and CPD [since these are equal angles].

8 Let O be the centre of a given circle. Construct two diameters EG and FH. At E and G draw tangents AEB and DGC. At F and H draw tangents BFC and AHD, cutting the

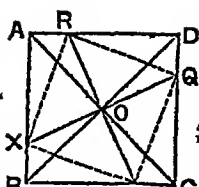


centre of a given circle—Draw any and FH. At E and G draw tangents AEB and DGC. At F and H draw tangents BFC and AHD, cutting the former tangents at

A, D, B, C. Then ABCD is the reqd. rhombus.

Proof—Because the \angle^s AHO, OFC are rt \angle^s (Theor. 46), therefore AD, BC are parallel (Theor. 13). Similarly, AB, DC are parallel. Hence the Fig. ABCD is a parallelogram, so that $AD = BC$ and $AB = DC$ (Theor. 21). But $AD + BC = AB + DC$ (See Ex. 14, page 177) $\therefore 2 BC = 2 DC$, or $BC = DC$. Hence the sides AB, BC, CD, DA are all equal to one another. Hence the fig. ABCD is a rhombus.

9. Let $ABCD$ and X a point on AB . Cons.—Draw the diagonals AC, BD intersecting at O . Produce it to meet CD at Q . Through O draw ROP perp. to AB and BC at P and R . Join RQ, QP, PX and XR . Then $XPQR$ is the reqd. square.

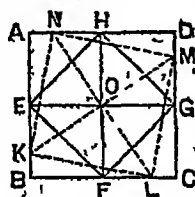


be a given square the side AB . diagonals AC, BD Join XO and produce it to meet CD at Q . Through O draw ROP perp. to AB and BC at P and R . Join RQ, QP, PX and XR . Then $XPQR$ is the reqd. square.

Proof—In the $\triangle^s AOR$ and OPC , because $AO=OC$ (Cor. 3, Theor. 21) the $\angle AOR = \angle POC$, and the $\angle OAR = \angle OCP$, \therefore the \triangle^s are congruent; so that $RO=OP$ (Theor. 17). Similarly it can be proved that $XO=OQ$. Now in the $\triangle^s XOR$ and RQO , $XO=OQ$, RO is common and the $\angle XOR = \angle RQO$ (being rt. \angle^s) therefore $XR=RQ$ (Theor. 4). Similarly it can be proved that $QP=PX$, $PX=XR$. Hence the fig. $XPQR$ is a rhombus.

The $\angle AOB = \angle ROX$ (being rt. \angle^s), take away the common $\angle AOX$. \therefore the $\angle XOB = \angle AOR$. Now, in the $\triangle^s BOX$ and AOR , $BO=AO$, the $\angle XOB = \angle AOR$ and the $\angle XBO = \angle RAO$ (each being 45°) $\therefore \triangle^s$ are equal. $\therefore OX=OR$ (Theor. 17). $\therefore \angle XRO = \angle RXO = 45^\circ$, since the third $\angle ROX$ of the $\triangle ROX$ is a rt. \angle . Also $OX=OP$ (since each $=OR$), $\therefore \angle OXP = \angle XPO = 45^\circ$. $\therefore \angle RXP = \angle RXO + \angle OXP = 90^\circ$. \therefore the rhombus $XPQR$ is a square.

10. Let $ABCD$ Cons.—Bisect the sides AB, BC, CD, DA at E, F, G, H respectively. Join EF, FG, GH, HE . Then $EFGH$ is the reqd square.



be a given square. sides AB, BC, CD, DA respectively. Join EF, FG, GH, HE . Then $EFGH$ is the reqd square.

Proof—Join FH and EG . Then FH and EG are equal and intersect at rt. \angle^s at O . That is the diagonals of

fig. EFGH are equal, and bisect one another at rt. \angle^s .
 \therefore the fig. EFGH is a square (see proof, Ex. 9).

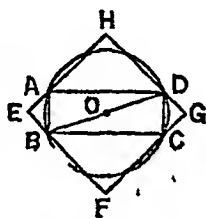
Let KLMN be any other inscribed square. Join the diagonals KM, NL; then these will intersect at the pt. O.

Because OK is greater than OE, and OL is greater than OF (Theor. 12), therefore KL^2 , which is $=OK^2 + OL^2$, is greater than EF^2 , which is $=OE^2 + OF^2$. That is, the sq. KLMN is less than the sq. EFGH. Similarly it can be proved that any other square inscribed in the given sq. ABCD, is greater than the sq. EFGH.

Hence EFGH is the square of minimum area inscribed in the given sq. ABCD.

11. Let ABCD

(i) Join BD, and describe a circle. are rt. angled tri- their common hypo- circle described on passes through the (page 165), and is therefore the circumscribed circle of the rectangle ABCD.



be a given rectangle on BD as diameter Since BAD and BCD angles and BD is tenuse, therefore the BD as diameter pts. A and C (Ex. 1,

(ii) Cons.—At the pts. A and D make the \angle^s DAH, ADH each $=45^\circ$, the arms AH, DH meeting at H. Then the $\angle AHD=90^\circ$. Through the pts. B and C draw st. lines EBF, FCG parallel to AH, DH respectively, meeting each other at F, and the st. lines HA, HD produced at the pts. E and G. Then the fig. EFGH is the reqd. square.

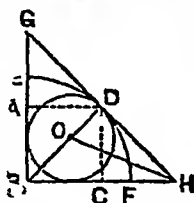
Proof—The fig. EFGH is a rectangle (by construction), therefore $EH=FG$, $HG=EF$ (Theor. 21).

Now, the $\angle HAD=45^\circ$, and the $\angle DAB=90^\circ$, therefore the $\angle EAB=45^\circ$. Consequently the $\angle EBA=45^\circ$. Hence EAB is an isosceles Δ . Similarly, it can be shown that DCG is an isosceles Δ .

Now in the Δ^s EAB and DCG, $AB=DC$, $\angle AEB=\angle DGC$ (being rt. \angle^s) and $\angle EBA=\angle GCD$ (each being 45°), therefore $EA=DG$ (Theor. 17). But the $\angle HAD=$

the $\angle HDA$ (by cons.), hence $HA = HD$ (Theor. 17).
Therefore $HE = HG$. Hence $HE = HG = FG = EF$.
Therefore the rect $EFGH$ is a square.

12. Let EBF
(i) Bisect the \angle
 BD meeting the arc
the tangent GDH
ing BE, BF produced
tively Bisect the
line HO meeting
inscribed circle of $\triangle GBH$.



be a given quadrant.
 EBF by the st. line
 EF at D . At D draw
to the arc EF meet-
at G and H respec-
 $\angle BHG$ by the st.
 BD at O . Draw the

Then O is the in-centre of the $\triangle GBH$ (Prob. 26).
Then the circle inscribed in the $\triangle GBH$ is the reqd circle,
because it touches each of the sides BG, BH , and touches
 GH at D . Now, since GH is a common tangent to
the circle and the arc EF at D , the circle touches the arc
 EF at D . Hence it is the reqd. circle.

(ii) From D draw DA, DC perp^s. to BG, BH respec-
tively. Then $ABCD$ is the reqd. square.

In the two $\triangle^s ABD, BCD$, because $\angle BAD = \angle BCD$
(being at angles, $\angle ABD = \angle DBC$ (by cons.) and BD is
common to both, \therefore the \triangle^s are identically equal
(Theor. 17) $\therefore AD = DC$. And the fig $ABCD$ is a rect-
angle (by cons.) Hence it is a square, and it is inscribed
in the quadrant EBF .

Page 200.

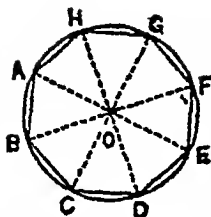
1. (i) With any
radius = 4 cm. des-
 COF be one of its
centre C and radius
cutting the circle at
 B and D draw the
 DOA . Join AB, BC ,



Since the $\triangle^s BOC, COD$ are equilateral, $\angle DOC = 60^\circ =$
 $\angle BOC$. $\therefore \angle AOC$ is also $= 60^\circ = \angle DOE = \angle EOF = \angle$
 FOA . Thus each of the \angle^s at $O = 60^\circ = \frac{1}{6}$ of 360° . $\therefore AB$
 DEF is the reqd. regular hexagon (Prob. 30).

pt. O as centre and
cribe a circle. Let
diameters. With
 $= CO$ draw an arc
 B and D . Through
diameters $BOE,$
 CD, DE, EF and FA .

(ii) With any radius = 4 cm. describe any two diameters angles to each other. AOE, GOC bisecting the first two diameters the angles at O is 360° . Join $AB, BC, CD, DE, EF, FG, GH$ and HA . Then $ABCDEFGH$ is the reqd. regular octagon (Prob. 30).



pt. O as centre and describe a circle. Draw BOF, HOD at rt. angles. Draw the diameters bisecting the angles between the first two diameters. Then each of the angles at O is evidently $= 45^\circ = \frac{1}{8}$ of 360° .

(iii) See fig. in Ex. (i).

Bisect the angles AOB, BOC , etc. at the centre O , by OH, OK, OL, OM, ON, OP respectively. Join $AH, HB, BK, KC, CL, LD, ME, EN, NF, FG$ and GA . Then each of the angles at $O = 30^\circ = \frac{1}{12}$ of 360° . \therefore the fig. $AHBKCLDMENFG$ is the reqd. regular dodecagon.

2 (i) With any radius = 1.5" describe a regular hexagon as in Ex. 1 (i); and let be its angular pts. the circle at these another at G, H, K , the resulting fig. $GHKLMN$ is the reqd. circumscribed regular hexagon.



pt O as centre and describe a circle. Inscribe a hexagon in this circle let A, B, C, D, E, F be its angular pts. Draw tangents to the circle meeting one another at L, M and N . The resulting fig. $GHKLMN$ is the reqd. circumscribed regular hexagon.

Join OH, OK, OL, OB and OC .

Proof—Because the $\angle^s OBK$ and OCK are rt. \angle^s , therefore the $\angle^s BOC$ and BKC together = 2 rt. \angle^s (Inf. 5, Theor. 16). But the $\angle BOC = 60^\circ$ [proved in Ex. 1, (i)], therefore the $\angle BKC = 120^\circ$. Similarly it can be proved that each of the $\angle^s CLD, DME, ENF, FGA, AHB = 120^\circ$. Hence the fig. $GHKLMN$ is equiangular.

Again because the circle touches the st. lines HK and KL , therefore OK bisects the $\angle HKL$ (Ex. 6, page 177).

Similarly OH, OL bisect the $\angle^s GHK, KLD$ respectively. Hence each of the $\angle^s OHK, OKH, OKL, OLK = 60^\circ$.

\therefore the $\triangle^s OHK, CKL$ are equiangular, and therefore equilateral. $\therefore HK = OK = KL$.

Similarly it can be proved that $KL = LM$, and so on. Hence the fig. $GHKLMN$ is also equilateral. Therefore $GHKLMN$ is a regular figure.

Measure all the sides of the hexagon $GHKLMN$ and they will be found to be equal to one another; also measure the angles and it will be found that each of the angles $= 120^\circ$. Hence the fig. is regular.

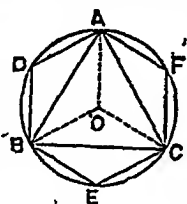
(22) With any radius $= 15''$ describe a regular octagon K, L, M, N, P, Q, R, S . Draw tangents to the circle at these pts. cutting one another at A, B, C, D, E, F, G and H . The resulting fig. $ABCDEFGH$ is the reqd circumscribed regular octagon.



Proof—Proceed as in the case of Ex. 2, (2).

Measure all the sides and angles of the octagon, and it will be found that all its sides are equal, and each of the angles $= 135^\circ$. Hence the fig. is regular.

3. Let O be the centre of a given circle. Inscribe a regular hexagon AD $BECF$ in it. Join AB, BC and CA . Then ABC is the inscribed equilateral triangle in it. Let a and b denote the lengths of their sides.



centre of a given circle. Inscribe a regular hexagon AD $BECF$ in it. Join AB, BC and CA . Then ABC is the inscribed equilateral triangle in it. Let a and b denote the lengths of their sides.

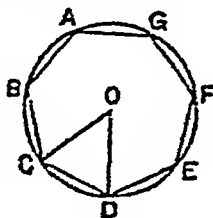
(2) Join OA, OB and OC . Then the $\angle BOC = 2$ the $\angle BAC$ (Theor. 38) $= 120^\circ$. $\therefore \angle^s OBC$ and OCB are together $= 180^\circ - 120^\circ = 60^\circ$ (Theor. 16). But $\angle OBC = \angle OCB$, since $OB = OC$, \therefore each of the $\angle^s = 30^\circ$. Also

the $\angle BEC$, being the angle of a regular hexagon $= 120^\circ$; and it can be proved as before that $\angle EBC = \angle ECB = 30^\circ$. Now, in the two $\triangle^s BOC, BEC$, side BC is common, $\angle OBC = \angle EBC$, $\angle OCB = \angle ECB$ (each being $= 30^\circ$). \therefore two \triangle^s are equal. \therefore the $\triangle BOC = \frac{1}{2}$ the fig. $BOCE$. Similarly it can be proved that $\triangle AOC = \frac{1}{2}$ the fig. $AOCE$, and the $\triangle AOB = \frac{1}{2}$ the fig. $AOBD$. Hence summing up we have the $\triangle ABC = \frac{1}{2}$ the hexagon $ADBECF$.

(ii) Because $\frac{1}{2} AB^2 = OB^2$ (Ex. 6, page 199), or $AB^2 = 3 OB^2$, and $OB = BE$,

$\therefore AB^2 = 3 AD^2$, that is, $a^2 = 3 b^2$

4. With any radius $= 2''$ describe a circle. At O make an $\angle COD$ nearly by means of the protractor. Join CD . Set off GA and AB each the circumference.



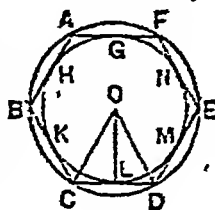
pt. O as centre and describe a circle. At O make an $\angle COD$ nearly by means of the protractor. Join CD . Set off GA and AB each the circumference.

$ABCDEFG$ is the reqd. inscribed heptagon.

Because 7 times the $\angle ABC + 360^\circ = 2 \times 7$ rt. $\angle^s = 1260^\circ$ (Cor. 1, Theor. 16), therefore the $\angle ABC = \frac{1}{7} (1260^\circ - 360^\circ) = 128\frac{4}{7}^\circ$. Measure the $\angle ABC$, and a side AB , and it will be found that $\angle ABC = 128^\circ 6'$ nearly, and $AB = 1.73''$.

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1. Draw a st. line $CD = 2''$, and at C and D make \angle^s 120° , making CB, DE each $= 2''$. At B and E again make the \angle^s CBA, FED each $= 120^\circ$, making BA, EF each $= 2''$. Join AF . Then $ABCDEF$ is the reqd. regular hexagon on a side of $2''$.



line $CD = 2''$, and at C and D make \angle^s 120° , making CB, DE each $= 2''$. At B and E again make the \angle^s CBA, FED each $= 120^\circ$, making BA, EF each $= 2''$. Join AF . Then $ABCDEF$ is the reqd. regular hexagon on a side of $2''$.

Bisect the \angle^s BCD, CDE by the st. lines CO, DO meeting at O . With centre O and radius OC describe a

circle; then this circle is the circumscribed circle of the hexagon $ABCDEF$ (Prob 31)

From O draw OL perp to CD . With centre O and radius OL describe a circle; then this circle is the inscribed circle of the hexagon $ABCDEF$ (Prob 31).

By calculation the $\angle OCD = 60^\circ = \angle ODC$, and hence $= \angle COD$ (Theor. 16). $\therefore \triangle OCD$ is equilateral. $\therefore OC = CD = 1''$, \therefore the circum-diameter $= 2''$. Now, $CL = \frac{1}{2} CD = \frac{1}{2}''$. $\therefore OL = \sqrt{OC^2 - CL^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = 1.73''$, therefore the in-diameter $= 3.46''$.

Measure the circum-diameter and the in-diameter, and they will be found to be $= 2''$ and $3.46''$ respectively.

2. See fig. in Ex 2, (i), page 200:—

Let O be the centre of the given circle, and let $ABCDEF$ and $GHKLMN$ be the inscribed and circumscribed regular hexagons. Join OH , OK , OB and OC . Let OK cut BC at P .

$$\text{Then } OP = \sqrt{OC^2 - CP^2} = \sqrt{OC^2 - \frac{1}{4} OC^2} = \frac{\sqrt{3}}{2} OC,$$

$$\text{and } OC = BC = \sqrt{OK^2 - KP^2} = \sqrt{OK^2 - \frac{1}{4} OK^2} = \frac{\sqrt{3}}{2} OK \\ = \frac{\sqrt{3}}{2} HK \quad \text{Therefore } HK = \frac{2}{\sqrt{3}} OC.$$

$$\text{The } \triangle OHK = \frac{1}{2} OB. HK = \frac{1}{2} OC \times \frac{2}{\sqrt{3}} OC = \frac{1}{\sqrt{3}} OC^2,$$

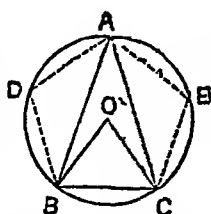
$$\text{and the } \triangle OBC = \frac{1}{2} OP. BC = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} OC \times OC = \frac{\sqrt{3}}{4} OC^2 \\ = \frac{1}{2} \frac{1}{\sqrt{3}} OC^2 = \frac{1}{2} \triangle OHK.$$

Now, the hexagon $ABCDEF = 6 \triangle OBC$, and the hexagon $GHKLMN = 6 \triangle OHK$

\therefore the hexagon $ABCDEF = \frac{1}{2}$ of the hexagon $GHKLMN$

If $OC = 10$ cm., then the area of the hexagon
 $ABCDEF = 6 \triangle OBC = 6 \times \frac{\sqrt{3}}{4} OC^2 = 6 \times \frac{\sqrt{3}}{4} \times (10)^2 =$
 $150\sqrt{3}$ or 259.8 sq. cm.

3. Let O be the centre of a given circle, and let $\triangle ABC$ be an isosceles triangle inscribed in it, the $\angle^s ABC, ACB = \angle BAC$. It is reqd. to shew that BC is a side of a regular pentagon inscribed in the circle.



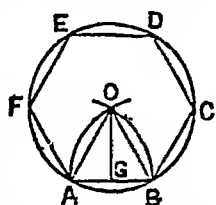
centre of a given circle, and let $\triangle ABC$ be an isosceles triangle inscribed in it, such that each of its angles is double of the $\angle BAC$. It is reqd. to shew that BC is a side of a regular pentagon inscribed in the circle.

The $\angle^s ABC + ACB + BAC = 180^\circ$ (Theor. 16), or
 $2 \angle BAC + 2 \angle BAC + \angle BAC = 180^\circ$.
 $\therefore \angle BAC = 36^\circ$. Join OB, OC . Then the $\angle BOC =$
 2 the $\angle BAC$ (Theor. 38) $= 72^\circ = \frac{1}{5}$ of 360° .

Hence BC is a side of a regular pentagon inscribed in the given circle (Prob. 30).

Note—See also Ex. 17, page 171.

4. (i) Draw a circle with centre O and radius $= 4$ cm. draw two arcs at O . With centre O and radius OA or OB describe a circle. CD, DE, EF each



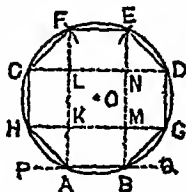
st. line $AB = 4$ cm, B , and radius $= 4$ cm. cutting one another at O and radius OA or OB . Set off chords BC, CD, DE, EF each equal to AB round

the circumference of the circle. Join FA . Then $ABCDEF$ is the reqd. hexagon [since $\triangle AOB$ is equilateral and therefore $\angle AOB = 60^\circ = \frac{1}{6}$ of 360°].

Area of the hexagon $ABCDEF = 6$ times the $\triangle OAB =$
 $6 \times \frac{\sqrt{3}}{4} AB^2$ (proved in Ex. 2) $= 6 \times \frac{\sqrt{3}}{4} \times 16 = 41.57$ sq. cm.

(ii) Draw a st. line $AB = 4$ cm. Produce it both ways to any pts. P and Q . At A and B draw AP, BQ perps. to AB . Bisect the $\angle^s FAP, EBQ$ by the st. lines AF, BE respectively, making each of them $= 1$ cm.

Draw HG, CD parallel to AF or BE making each = 4 cm. With centres G and D, and radius = 4 cm. draw two arcs cutting the line AF at F and BE at E. Join GF, DE and EFGH is the reqd. octagon (since each angle = 135°).



Join GD cutting AF at L, BE at N. Join HC cutting AF at K and BE at M. Then the octagon is divided into 4 rt. angled isosceles triangles, four rectangles and a central square.

Now $AH^2 = AK^2 + HK^2 = 2 AK^2$. Therefore $AK = \frac{AH}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$ cm.

\therefore Area of the octagon = $4 \Delta AHK + 4 \text{ rect } ABAK + KM^2 = 4(\frac{1}{2} HK \cdot AK) + 4(AB \cdot AK) + AB^2 = 4 \times (\frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2}) + 4 \times (4 \times 2\sqrt{2}) + 4^2 = 16 + 32\sqrt{2} + 16 = 77.25 \text{ sq. cm.}$

Page 202.

1. We know that $\pi = \frac{\text{circumference}}{\text{diameter}}$; \therefore in case (i)

$\pi = \frac{16}{5.1} = 3.13725$; in case (ii) $\pi = \frac{8.8}{2.8} = 3.14286$; and in

case (iii) $\pi = \frac{13.5}{4.3} = 3.13953$ And mean of the three

results = $\frac{3.13725 + 3.14286 + 3.13953}{3} = 3.13988$.

2. Length required for 20 complete turns = $75.4''$.

\therefore ... 1 ... turn = $3.77''$.

Hence the circumference = $3.77''$. $\therefore \pi = \frac{3.77}{1.2} = 3.1417$ nearly.

3. The wheel makes 400 revolutions in 977 yards.
 $\therefore \dots \dots \dots$ 1 revolution $\dots \dots 2.4425$ yds.
 Hence the circumference = 2.4425 yds.

$$\therefore \pi = \frac{2.4425 \text{ yds.}}{28 \text{ in.}} = \frac{2.4425 \times 3 \times 12}{28} = 3.140357.$$

Page 205.

1. The circumference of a circle = $2\pi r$, \therefore in case (i) the circumference = $2 \times 3.14 \times 4.5 = 28.3$ cm., and in case (ii) the circumference = $2 \times 3.1416 \times 100 = 628.3$ cm.

2 The area of a circle = πr^2 ; \therefore in case (i) the area = $3.1416 \times (2.3)^2 = 16.62$ sq. in., and in case (ii) the area = $3.141593 \times (10.6)^2 = 352.99$ sq. in.

3. See fig in Ex. 1, Page 199.

Let ABCD be the circle inscribed in the square EFGH whose side = 3.6". \therefore The radius BO = $\frac{1}{2}$ BD = $\frac{1}{2}$ EH = 1.8 cm.

Hence the circumference = $2\pi r = 2 \times 3.1416 \times 1.8 = 11.31$ cm.

And area = $\pi r^2 = 3.1416 \times (1.8)^2 = 10.18$ sq. cm.

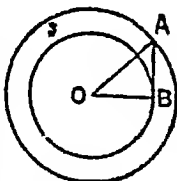
4. See fig. in 1, Page 199.

Since the diameter of the circle is the diagonal of the square. \therefore the diagonal = $2 \times 7 = 14$ cm. And the area of the square = $\frac{1}{2}$ (product of diagonals) = $\frac{1}{2} \times 14 \times 14 = 98$

sq. cm. And the area of the circle = $\pi r^2 = \frac{22}{7} \times 7^2 = 154$

sq. cm. \therefore the difference of the areas = $154 - 98 = 56$ sq. cm.

5. Let O be the common centre of two concentric circles of radii 5.7" and 4.3". Then the area of the circular ring between these two circles = $\pi(OA^2 - OB^2)$



= $3.1416 \times 14 = 43.98$ sq. in.

common centre of two radii 5.7" and 4.3", circular ring between $OA^2 - OB^2 = \pi(OA^2 - OB^2)$ (5.7 \times 5.7 - 4.3 \times 4.3)

6. See fig in Ex 5.

Let O be the centre of two concentric circles, and let AB be drawn tangent to the inner circle from any point A on the outer circle. Area of a circle of radius $AB = \pi \cdot AB^2 = \pi (OA^2 - OB^2) = \text{area of the ring (see Ex. 5)}.$

7. See fig. in Ex. 1, Page 199.

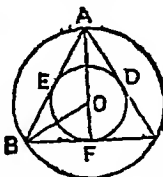
Let $ABCD$ be the rectangle inscribed in a circle. Join AC, BD . The area of the rectangle $= AB \times AD = 8 \times 6 = 48$ sq. cm. The diameter BD of the circle $= \sqrt{AB^2 + AD^2} = \sqrt{64 + 36} = 10$ cm. \therefore The radius $= 5$ cm. Hence the area of the circle $= \pi r^2 = 3.1416 \times 5 \times 5 = 78.5$ sq. cm. \therefore The area of the four segments outside the rectangle $= 78.5 - 48 = 30.5$ sq. cm.

8. The area of the reqd. square $=$ the area of the circle whose radius is $5'' = \pi \times 5^2 = 3.1416 \times 25 = 78.54$ sq. in. \therefore the side of the required square $= \sqrt{78.54} = 8.86'' = 8.9''.$

9. See fig. in Ex 5.

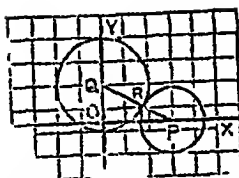
Let x'' be the radius of the smaller circle. Then the radius of the greater circle $= (x + 1)''$. \therefore The area of the ring $= \pi(x + 1)^2 - \pi x^2 = \frac{22}{7}(2x + 1) = 22$ (given). $\therefore x = 3$. \therefore the radii are $4''$ and $3''$.

10 Let ABC be the equilateral triangle whose side $= 4''$ and let ABC and EDF be the cir—



scribed circles. Then these circles are concentric, h F^2 av their common centre at O . $BF = \frac{1}{2} BC = 2''$. $\therefore AF = \sqrt{AB^2 - BF^2} = \sqrt{16 - 4} = 2\sqrt{3}$ in. $\therefore AO = \frac{2}{3} AF = \frac{2}{3} \times 2\sqrt{3} = \frac{4}{3}\sqrt{3}$ in, and $OF = \frac{1}{3} AF = \frac{2}{3}\sqrt{3}$ in. \therefore The difference of the areas of these two circles $= \pi(AO^2 - OF^2) = 3.1416 \times \left(\frac{16}{3} - \frac{4}{3}\right) = 12.57$ sq. in.

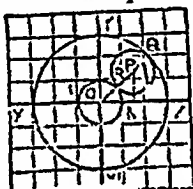
11. Let P points $(1.5'', 0)$ and



and Q be the points $(0, .8'')$ respectively.

Join QP . Then $QP = \sqrt{OP^2 + QP^2} = \sqrt{(1.5^2) + (.8)^2} = 1.7''$. With centres P and Q and radii $= .7''$ and $1.0''$ draw two circles; then they will touch each other externally, because the sum of their radii $= .7'' + 1.0'' = 1.7'' =$ the distance between the centres P and Q . \therefore their circumferences are $= 2 \times 3.14 \times .7 = 4.4''$, and $2 \times 3.14 \times 1 = 6.3''$ nearly. And their areas are $= 3.14 \times (.7)^2 = 1.54$ sq. in.; and $3.14 \times 1^2 = 3.14$ sq. in. nearly.

12 Let P be With P as centre describe a circle. Join circle at R and pro-



duce OR to meet it the point $(1.6'', 1.2'')$ and radius $= 1''$ describe a circle. Join OP cutting the circle at R and produce OR to meet it

again at Q . From P draw PM perp to OX . Then $OP = \sqrt{PM^2 + OM^2} = \sqrt{(1.6)^2 + (1.2)^2} = 2''$. $\therefore OR = OP - PR = 2'' - 1'' = 1''$, and $OQ = OP + PQ = 2'' + 1'' = 3''$. Therefore the circles described with centre O and radii $1''$ and $3''$ will touch the first circle, the former *externally* at R , and the latter *internally* at Q . Draw the circles as shown in the figure.

Page 206

1. See fig in Ex 8 page 189.

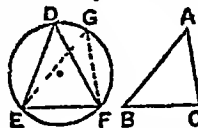
Let AB and CD be any two parallel st lines and EF any other st. line meeting them. It is reqd. to describe circles to touch AB , CD , EF .

(i) Locus of the centres of circles touching AB and EF is one or other of the lines EO , EP which bisect the angles AEF , BEF respectively [Note VI, page 188].

(ii) Locus of the centres of circles touching CD and EF is one or other of the lines FO and FP bisecting the angles CFE and DFE respectively. [Note VI, p 188] \therefore The points O and P , where these st. lines intersect, are the

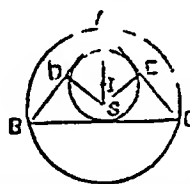
centres of the required circles (iii) Again, the centres of all circles touching two parallel straight lines lie on a line parallel to the given lines and midway between them. \therefore the points O and P are equally distant from CD , hence the radii of the two circles are equal \therefore the two circles are equal.

2. Let ABC and DEF be two triangles which have their bases BC , EF equal, and the vertical $\angle EDF$. It is reqd. to show



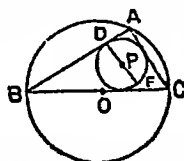
that their circumcircles are also equal. Place the $\triangle ABC$ over the $\triangle DEF$ such that the pt B falls on the pt E , and BC along EF , then because $BC=EF$, C will coincide with F . Let EGF represent the new position of the $\triangle ABC$. Now, since the $\angle EGF = \text{the } \angle EDF$, the points D, G, F, E are concyclic [Converse, Theor 39]. \therefore the circum circle of the $\triangle DEF$ is also the circum-circle of the $\triangle EGF$. Therefore the circum circles of the $\triangle DEF$ and ABC are equal.

3. Let ABC be a triangle, and let S and I be its circum-centre and in-centre. And let S lie on AI . It is reqd. to prove that $AB = AC$. Because I is the in-centre, \therefore the $\angle BAI = \text{the } \angle CAI$.



draw SD , SE perp. to AB , AC . Then since, S is the circumcentre, $\therefore D$ and E are the mid. pts. of AB and AC respectively (Prob. 25). In the $\triangle SAD$ and SAE , the $\angle SDA = \text{the } \angle SEA$ being rt. \angle s, the $\angle SAD = \text{the } \angle SAE$, and AS is common to both \therefore the \triangle s are equal in all respects [Theor 17]. $\therefore AD = AE$. And since AD , AE are halves of AB , AC respectively, $AB = AC$.

4. Let ABC be a triangle rt. \angle at A , let D , d denote the diameters of the inscribed and



of the inscribed and

circles. It is reqd. to show that $D + d = b + c$.

note the diameters

the circumscribed

Area of the $\triangle ABC = \frac{1}{2} (a+b+c)r$; where r = radius of the inscribed circle [Ex. 5, p. 198], and it is also $= \frac{1}{2} cb$. $\therefore \frac{1}{2} cb = \frac{1}{2} (a+b+c)r$, $r = \frac{cb}{a+b+c}$. \therefore

$$d = 2r = \frac{2cb}{a+b+c}$$

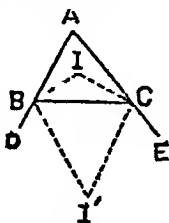
Again because the $\angle A$ is a rt. angle, $\therefore a^2 = c^2 + b^2$, and $D = CB = a$ [Prob. 10]. $\therefore D + d = a + \frac{2cb}{a+b+c} = \frac{ac+ab+a^2+2rb}{a+b+c} = \frac{a(c+b)+c^2+b^2+2cb}{a+b+c} = \frac{a(c+b)+c^2+b^2+2cb}{a+b+c} = \frac{(c+b)(a+b+c)}{a+b+c} = c+b$.

5. See fig in Ex 5 page 198.

Let the inscribed circle of a $\triangle ABC$ touch the sides AB, BC, CA at D, E and F respectively. It is reqd. to prove that the angles of the $\triangle DEF$ are respectively $90^\circ - \frac{1}{2}A$, $90^\circ - \frac{1}{2}B$, $90^\circ - \frac{1}{2}C$. Because AF and AD are two tangents drawn from A, $\therefore AF = AD$ [Cor., Theor. 47]. \therefore the $\angle AFD = \angle ADF$. Now the $\angle FAD + \angle ADF + \angle AFD = 180^\circ$, that is $2 \angle ADF + \angle A = 180^\circ$. $\therefore \angle ADF + \frac{1}{2}A = 90^\circ$. \therefore the $\angle ADF = 90^\circ - \frac{1}{2}A$.

But the $\angle ADF = \angle DEF$ in the alt. segment [Theor. 49]. \therefore the $\angle DEF = 90^\circ - \frac{1}{2}A$. Similarly, it can be proved that the $\angle DFE = 90^\circ - \frac{1}{2}B$, and the $\angle FDE = 90^\circ - \frac{1}{2}C$.

6. Let ABC be a triangle and let I, I' be the centres of the the escribed circle. It is reqd. to prove are concyclic. Be- are the internal and C (prob. 26), and



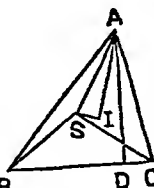
external bisectors of the $\angle^s B$ and C [Prob. 27]. \therefore the $\angle^s IBI'$ and ICI' are rt. \angle^s . $\therefore \angle IBI' + \angle ICI' = 2 \text{rt. } \angle^s$. \therefore the points I, B, I' and C are concyclic [converse, Theor. 40].

7. See fig. in Ex. 5, page 198.

Let ABC be a triangle, and let the inscribed circle touch the sides AB, BC, CA at D, E, F, respectively.

It would be sufficient, if we prove that $AC - AB = CE - BE$. Because $AF = AD$, $BE = BD$, and $CF = CE$ [Cor. Theor. 47] $AC - AB = (AF + CF) - (AD + BD) = (AD + CE) - (AD + BE) = CE - BE$.

8. Let ABC be
the side AB is greater
 I and S be its incen-

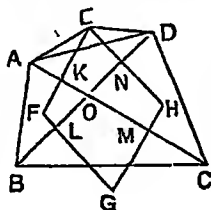


a triangle, of which
than AC , and let
tre and circumcentre.

Join IS , AI and AS . It is reqd. to prove that the $\angle IAS = \frac{1}{2}(\angle C - \angle B)$. Join SB , SC . Since $SB = SC$ (each being circum-radius), \therefore the $\angle SBC = \angle SCB$. Similarly $\angle SBA = \angle SAB$ and $\angle SCA = \angle SAC$. $\therefore \angle C - \angle B = (\angle ACS + \angle BCS) - (\angle ABS + \angle CBS) = \angle ACS - \angle ABS$ [since $\angle CBS = \angle BCS$], $= \angle CAS - \angle BAS = (\angle CAI + \angle IAS) - (\angle BAI - \angle IAS) = 2$ the $\angle IAS$ [$\because \angle CAI = \angle BAI$]. \therefore the $\angle IAS = \frac{1}{2}(\angle C - \angle B)$.

(11) From A draw AD perp. to BC . Then since IA is the bisector of the $\angle BAC$, $\therefore \angle DAI = \frac{1}{2}(\angle C - \angle B)$. [Ex 3, p. 138], \therefore the $\angle DAI =$ the $\angle IAS$, i. e., AI is the bisector of the $\angle DAS$.

9. Let $ABCD$
Join diagonals AC ,
 O Bisect AO , BO
the pts. K , L , M
Through these pts.
 FLG , GMH ,
 AO , BO , CO , DO



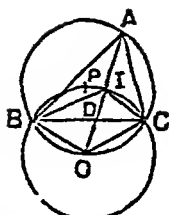
be a quadrilateral.
 BD intersecting at
 CO and DO , at
 N respectively.
draw st lines EKF ,
 HNE perps. to
respectively; and

let them meet at the pts. E , F , G , H as in the fig. Then E , F , G , H are the circumcentres of the Δ^s AOD , AOB , BOC and COD respectively [Prob. 25]

It is reqd. to prove that $EFGH$ is a parallelogram

Because EF , GH are both perps. to AC , therefore EF is parallel to GH (Ex 2, page 41). Again because EH , FG are both perps. to BD , therefore EH and FG are parallel (Ex. 2, p. 41). Hence the fig $EFGH$ is a parallelogram.

10. Let ABC be a triangle and let I be the centre of the inscribed circle. Circumscribe a circle about the $\triangle ABC$ and let P be its centre (Prob. 25). Join AI and produce it to meet the circum-circle at O . Join BI , IC .



It is reqd. to prove that O is the centre of the circle circumscribed about the $\triangle BIC$. Join BO , CO .

Proof—Because I is the in-centre, therefore AI , BI and CI bisect the \angle^s BAC , ABC and ACB respectively (Prob. 26). Therefore the $\angle OIC = \text{the } \angle IAC + \text{the } \angle ICA$ (Theor. 16, Obs.) $= \frac{1}{2} \angle BAC + \frac{1}{2} \angle ACB$. Again because the $\angle OCB = \text{the } \angle OAB$ (Theor. 39) $= \frac{1}{2} \angle BAC$, and the $\angle BCI = \frac{1}{2} \angle ACB$, therefore the $\angle OIC = \text{the } \angle OCB + \text{the } \angle BCI = \text{the } \angle OCI$, $\therefore OC = OI$ (Theor. 6). Likewise it can be proved that $OB = OI$. Therefore $OB = OI = OC$. Hence O is the centre of the circle circumscribed about the $\triangle BIC$ (Theor. 33).

11. Let BC be the base, GH the altitude, and KL the radius of the circumscribed circle of a triangle. It is reqd.

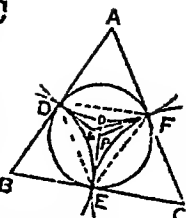


to construct the triangle—

Cons. Bisect BC at D . At D draw DF perp. to BC . Then the circum-centre lies on DF (Prob. 25). With centre C and radius $= KL$ draw an arc cutting FD at O . With centre O and radius OC draw the circle BAC . At B draw BE perp. to BC making $BE = GH$. From E draw EA' parallel to BC , cutting the circle at A and A' . Join AB , AC , $A'B$ and $A'C$.

Then ABC and $A'BC$ are the two reqd. triangles satisfying the given conditions.

12. The pts. A, line; also A, F, C and B, E, C are in 48). That is, the sides of the \triangle draw DP, EP and let them meet are tangents at D,



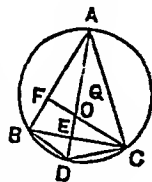
D, B are in one st. line; are in one st. line; one st. line (Theor. p's D, E, F lie on ABC. At D, E, perps. to AB, BC, at P. Then DP, EP E Join PF. If PF

is not perp to AC, let any other line FQ be perp. to AC meeting EP produced at O and -P at Q. PD, PE are tangents to the same circle from P. \therefore PD = PE (Cor. Theor. 47). For the same reason OE = OF. and QD = QF. Now QD = QF = QO + OF = QO + OE = QO + OP + PE = QO + OP + PD = QO + OP + PQ + QD, which is absurd. Hence PF is perp. to AC, and therefore tangent at F, \therefore by Cor. Theor. 47, PD = PE = PF.

\therefore a circle drawn with centre P and radius = PD, must pass through the pts. E and F, and also must touch the sides AB, BC, CA, at D, E, F (\because radii PD, PE, PF = are perps. to the sides); i.e., the circle is circumscribed circle of the $\triangle DEF$, also is the inscribed circle of the $\triangle ABC$.

Page 209.

1. Let O be or C, and let the perp. the circumcircle at prove that OE = ED.



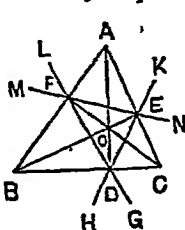
thocentre of the $\triangle AB$ AE (from A) meets D It is reqd. to

Join CO and produce it to meet AB at F. Join CD. Since $\angle AEC$, $\angle FC$ are rt. \angle s, $\therefore \angle OCE = 90^\circ - \angle EOC$. and $\angle OAF = 90^\circ - \angle AOF$, because $\angle AOF = \angle EOC$, their complements are equal, i.e., $\angle OAF = \angle OCE$. \therefore the $\angle DCB = \angle DAB$ (Theor. 39) = $\angle OCE$. Now in two \triangle s OCE, DCE,

because $\begin{cases} \angle OEC = \angle DEC \text{ (being rt. } \angle^s), \\ \angle OCE = \angle DCE \text{ (proved),} \\ EC \text{ is common to both,} \end{cases}$

\therefore two \triangle^s are identically equal ; $\therefore OE = ED$.

2. (i) Let ABC triangle. Draw AD , AE , BF , CG on opp. sides. Then DEF is the pedal \triangle . It is reqd. to prove that AB , BC , CA are external bisectors of the $\angle^s F$, D , E of the pedal \triangle .

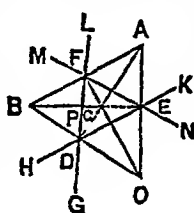


be an acute-angled \triangle . Draw AD , BE , CF , perps from A , B , C on opp. sides. Join DE , EF , FD . It is reqd. BC , CA are external bisectors of the $\angle^s F$, D , E of the pedal \triangle .

FC is the internal bisector of the $\angle DFE$. (Theor. II, page 208), and AB is perp to FC . \therefore it is the external bisector of the same $\angle DFE$, because internal and external bisectors of an angle are at rt. angles to one another (See Ex. 6. page 13).

Similarity it can be shown that BC , CA are external bisectors of $\angle^s FDE$ and DEF respectively.

(ii) Let ABC angled at C . Draw from A , B , C to opp. sides. E will be pts. on BC produced respectively. Join DE , EF , FD and produce them both ways. Then DEF is the pedal \triangle .



be a \triangle obtuse-angled at C . Draw AD , BE , CF perps from A , B , C to opp. sides. Then D and E will be pts. on BC produced respectively. Join DE , EF , FD and produce them both ways. Then DEF is the pedal \triangle .

Now, CF bisects the $\angle DFE$ internally \therefore AB (being perp to CF) is the external bisector of the $\angle DFE$.

Again, AE bisects the $\angle FEK$ (Theor. II on p. 208), i. e., bisects the $\angle FED$ externally, (Note at the bottom of p 208) \therefore ECF , being perp. to AE bisects the $\angle FED$ internally. For the same reason DCA being p-rp. is the external bisector BD of the $\angle FDE$, bisects the $\angle FDE$ internally.

3. See figs in Ex. 2 — First let us suppose the $\triangle ABC$ to be acute-angled as in Fig. in Ex. 2 (i). The $\angle BOC = \angle FOE$. The angles $\angle AFO$, $\angle AEO$ of the quad.

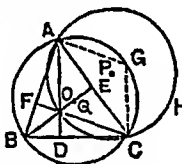
$\angle FOE$ are supplementary (since each is a rt. \angle). \therefore the fig. is concyclic (Converse, Theor. 40). $\therefore \angle^s FOE$ and $\angle^s FAE$ are supplementary. But $\angle FOE = \angle BOC$, $\therefore \angle BOC$ and $\angle FAE$ (i.e. the $\angle BAC$) are supplementary.

Let the Δ be obtuse-angled at C , as in Fig. in Ex 2 (11). Since $\angle ODA = \angle OFA$ being rt. \angle^s , the pts. F, A, O, D are concyclic (Converse, Theor. 39). $\therefore \angle FAD = \angle FOD$ in the same segment FAD of the circle (Theor. 39), i.e., the $\angle BAC = \angle BOC$.

4. See fig. in Ex 2 (2)—In the ΔBOC , the lines BF, OD are perps. from vertices B, C, O to opp. sides CO, BO, BC , and they intersect at A . Hence A is the orthocentre of the ΔBOC .

Similarly, it can be proved that B is the orthocentre of the ΔAOC , and C is that of the ΔAOB . And O is given to be the orthocentre of ΔABC . \therefore each of the four pts O, A, B, C is the orthocentre of the Δ whose vertices are the other three.

5. Let O be the ΔABC . Join OA , scribe circles about AOC, AOB . It is all these circles are



orthocentre of the ΔBOC . Circumscribe the $\Delta^s ABC, BOC$ and reqd to prove that equal.

Take any pt. G on the circle circumscribing the ΔABC on the side of AC remote from B . Join AG, CG .

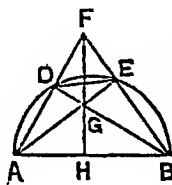
The $\angle BOC =$ supplement of the $\angle ABC$ [Ex. 3 (2), = the $\angle AGC$ (Theor. 40)]. Now fold the fig. $AOCG$ about the st. line AC , then the pt. G coincides with a pt., say G' on the same side of AC as O , and AG, CG coincide with AG', CG' . Now $\angle AOC = \angle AGC = \angle AG'C$, $\therefore C$ and G' lie on the same arc AOC (Converse, Theor. 39). That is, the pt. G on the arc AGC coincides with a pt. G' on the arc AOC . By taking other pts. on the arc AGC , it can be similarly shown that each of them coincides with corresponding pts. on the arc AOC . \therefore the whole arc AGC coincides with the arc AOC . \therefore the segment $AGC =$ the segment AOC .

Similarly by taking any pt., say K , on the arc AHC and joining AK , KC , it can be proved that segment ABC = segment AHC .

\therefore by adding the circle $ABCG$ = the circle AOC .

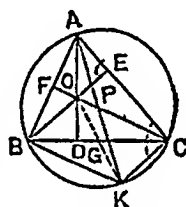
In the same way it can be proved that each of the circle circumscribing $\triangle^s AOB$, BOC is also = the circle $ABCG$.

6. Each of the a rt. \angle (Theor. 41), perps. from B and A of the $\triangle AFB$. $\therefore G$ the $\triangle AFB$. Now the must pass through G , AB .



$\angle^s ADB$ and AEB is \therefore , BD and AE are on opp. sides AF and is the orthocentre of perps. from F on AB $\therefore FGH$ is perp. to

7. Let ABC be a \triangle CF perps. to AC , AB , at O . Then O is the crite a circle about draw the diameter Then $BOCK$ shall be



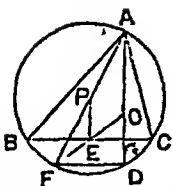
triangle. Draw BE cutting one another orthocentre. Describe the $\triangle ABC$, and AK . Join BK , CK . a parm.

Since $\angle ACK$ is a rt. \angle (Theor. 40), $\angle A, CK' = \angle AEB$. $\therefore BE$ $i. e.$ BO and KC are parallel (Theor. 13). Similarly it can be proved that CF , $i. e.$ CO and BK are parallel. \therefore fig. $BOCK$ is a parallelogram.

8. See Fig. in Ex. 7—Let ABC be a \triangle . Draw BE , CF perps. from B , C on AC , AB , and let them cut at O . Then O is the orthocentre of $\triangle ABC$. Describe a circle about the $\triangle ABC$, and draw the diameter AK . Bisect BC at G . Join OG , and produce it. It is reqd. to prove that it will pass through K . Join OK .

Now $BOCK$ is a parm. (proved in Ex. 7). \therefore its diagonals BC , OK bisect one another. That is, the pt. G , the mid. pt. of BC , lies on OK . \therefore pts. O , G , K , are in same st. line; $i. e.$, OG produced passes through K . Also $OG = GK$,

9. Let O be the orthocentre of a $\triangle ABC$. Draw AG perp. from A to BC . Then O lies on AG . Describe a circle about the $\triangle ABC$. Bisect base BC at E . Join OE , produce OE and AG to meet the circumcircle at F and D . Join DF . It is reqd. to prove that DF is parallel to the base BC . Join AF . Then AF is diameter through A (See Ex. 8). $\therefore \angle ADF = 1 \text{ rt } \angle$ (Theor. 41) $= \angle AGB$. $\therefore BC$ and FD are parallel (Theor. 13).

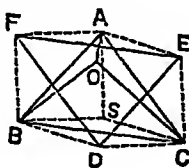


orthocentre of a \triangle perp. from A to BC . Describe a circle about the $\triangle ABC$. Bisect base BC at E . Join OE and AG to meet the circumcircle at F and D . Join DF . It is reqd. to prove that DF is parallel

10. See Fig. in Ex. 9—Let O be the orthocentre and P the circumcentre of a $\triangle ABC$. Draw AOG perp. from A to BC . Describe the circle about the $\triangle ABC$, and draw the diameter APF . Join OF cutting BC at E . Then E is the mid. pt. of BC as well as of OF (proved in Ex. 8). Join PE . Then PE is perp. to BC from E (Theor. 31). It is reqd. to prove that $AO = 2 PE$.

In the $\triangle AFO$, P is the mid. pt. of AF , and E the mid. pt. of OF . $\therefore PE = \frac{1}{2} AO$. or $AO = 2 PE$. (Ex. 3, page 64)

11. Let O be a $\triangle ABC$. Join S, D, E, F be the the $\triangle^s ABC, BOC$, tively. Join DE , to prove that the DEF in all respects



the orthocentre of OA, OB, OC . Let circumcentres of AOC, AOB respectively EF, FD . It is reqd. $\triangle ABC =$ the \triangle

Join $SA, SB, SC, FA, FB, EA, EC, DB$ and DC . These are the radii of circles circumscribed about the $\triangle^s ABC, BOC, AOC$ and AOB which are all equal (proved in Ex. 5). \therefore these lines are all equal to one another.

\therefore each of the figs. $SBDC, SAFB$ and $SAEC$ is a rhombus.

$\therefore CD$ is parallel to BS , and BS is parallel to AF . $\therefore CD$ and AF are parallel, and they are also equal $\therefore AC = FD$ (Theor. 20).

Similarly it can be shown that $AB = ED, BC = FE$. Thus we have the three sides of the $\triangle ABC$ respectively

equal to the three sides of the $\triangle DEF$, \therefore the \triangle^s are equal in all respects.

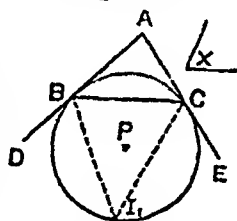
12. See Fig. in Ex. 9.—Let A be the given vertex, O the orthocentre, and P the circumcentre. It is reqd. to construct the triangle.

The triangle is constructed if we know the base. Now from Ex. 10, we know that AO is double the perp. distance of the base from P , and is parallel to that perp. Hence we have the following construction.

Construction—Join AO , AP . With P as centre and radius PA draw the circle $ACDB$. From P draw PE parallel to AO , making $PE = \frac{1}{2} AO$. At E draw BEC perp to PE meeting the circle at B and C . Join AB , AC . Then ABC is the reqd. triangle.

Page 211.

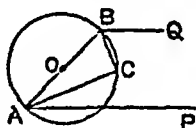
1. Let BC be the given vertical line. ABC be one of the \triangle^s whose vertical angle is $\angle X$. Produce AB to any pt. D . Bisect BC and BCE by BI_1 . Then I_1 is the excentre opp. to A . It is reqd. to find the locus of I_1 .



the given base and vertical angle; and let \triangle^s on the base have angle $A = \angle X$. pt D and AO to the ext. \angle^s CBD CI_1 intersecting at

The $\angle BI_1C = 90^\circ - \frac{1}{2} A$ (See Ex. 7, p 47) = constant, since $\angle A$ is constant (being always $= \angle X$), and BC is a given line. \therefore locus of I_1 is the arc of a segment of which BC is a chord, and which contains an angle $= 90^\circ - \frac{1}{2} A$.

2. Let AB be the given st. line; let AP , BQ be any two parallel st. lines drawn through A and B respectively. Bisect the \angle^s PAB , QBA by them meet at C . It is reqd. to find the locus of C .



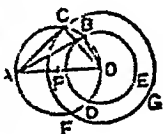
the given st. line; two parallel st. lines and B Bisect the \angle^s AC , BC , and let it is reqd. to find the

The sum of the \angle^s PAB and $QBA = 180^\circ$ (Theor. 14). \therefore sum of their halves $= 90^\circ$, i. e., $\angle ABC + \angle BAC = 90^\circ$ \therefore the $\angle ACB = 90^\circ$.

\therefore the locus of C is the circle described on AB as a diameter (Theor. 41).

3. See Ex 6 page 165.

4. Let BDE be concentric circles is O . Let A be the be a tangent drawn BDE It is reqd. to pt. B .



one of the system of whose common centre fixed pt., and let AB from A to the circle find the locus of the

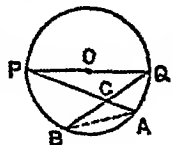
Join OB, OA . Since O and A are fixed pts., OA is a fixed st. line. And the $\angle ABO$ is a rt. \angle (Theor 46). \therefore locus of B is the circle drawn on OA as diameter.

5 See fig in Ex 7, page 170. Let $BCDE$ be the given circle, and D and E two fixed pts. on it. Let DC, EB be two such st. lines drawn from D and E , that the arc BC intercepted between them be of constant length, and let them meet at A . It is reqd. to find the locus of A .

Since arcs DE and BC are of constant lengths, the \angle^s DBE and BDC subtended by these arcs at the circumference are also of constant magnitudes.

Now the $\angle DAB = \text{the } \angle BDC - \text{the } \angle ABD$ (Theor. 16). \therefore the $\angle DAB$ or the $\angle DAE$ is also constant, and since the $\angle DAE$ stands on the fixed line DE , \therefore the locus of A is the arc of a segment of which DE is a chord, and which contains angle $= \text{the } \angle BDC - \text{the } \angle ABD$.

6. Let A, B be circumference of a PQ be any diameter. them intersect at C locus of C .

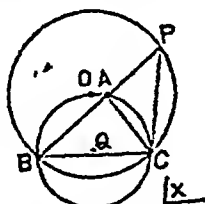


two fixed pts. on the circle $ABPQ$, and let Join AP, BQ and let It is reqd. to find the

Join AQ . Since A, B are fixed pts. arc AB is of some fixed length \therefore the $\angle AQB$ subtended by this arc at the circumference is of constant magnitude. And the $\angle PAQ$ is a rt. \angle (Theor. 40). \therefore the $\angle ACB$ which

$= \angle PAQ + \angle AQB$ (Theor. 16) is also constant. And since the $\angle ACB$ stands on a fixed line AB , the locus of C is the arc of a segment of which AB is a chord, and which contains an angle $= 90^\circ + \text{the } \angle AQB$

7 Let BAC be described on the fixed its vertical $\angle BAC$ $\angle X$. Let BA be that $BP = BA + AC$. the locus of P . Join



any triangle describe BC , and having equal to the given produced to P such it is reqd to find PC .

Since $BP = BA + AC$, therefore $AP = AC$, and hence the $\angle APC = \text{the } \angle ACP$ (Theor. 5). The $\angle BAC = \text{the } \angle APC + \text{the } \angle ACP$ (Theor. 16, Obs.) $= 2 \text{ the } \angle APC$. Therefore the $\angle APC = \frac{1}{2} \text{ the } \angle BAC = \frac{1}{2} \angle X$. Hence the $\angle APC$ is also constant. Therefore the locus of P is the arc of a segment on the fixed chord BC , containing an angle $= \frac{1}{2} \text{ the } \angle X$

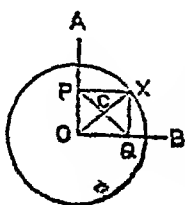
8. Let CBA be which AB is the any other chord AC complete the parallelo-



the diagonals DA, CB cutting one another at O . It is reqd. to find the locus of O .

Since the diagonals of a parallelogram bisect one another, therefore O is the middle pt. of the chord BC ; and since this chord passes through the fixed pt B , therefore the locus of its middle pt. O is the circle OBQ , whose diameter $BQ = \text{the radius of the given circle } CBA$ (See Ex. 6, page 165)

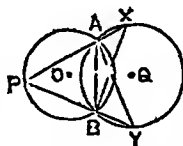
9 Let OA, OB at rt. angles to one be a position of the slides between them. PX, QX perps. to let the perps. meet at the locus of X .



be two rulers placed another, and let PQ straight rod which From P and Q draw OA and OB , and X It is reqd. to find

The fig. $POQX$ is by construction, a rectangle; therefore its diagonals OX , PQ are equal. Since the rod PQ is of constant length $\therefore OX$ is also of constant length, and the pt. O is a fixed pt. Therefore the locus of X is the quadrant intercepted between OA and OB , of the circle whose centre is O , and whose radius = the length of the rod PQ .

10. Let two circles intersect at A and B , and let P be any pt. on the circumference of one of them. From P two st. lines PA , PB are drawn to cut the other circle at Y and X . Join AY , BX intersecting at C . It is reqd. to find the locus of C .

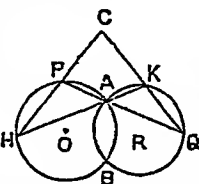


cles intersect at A and B . From P two st. lines PA , PB are drawn and produced to cut the other circle at Y and X . Join AY , BX intersecting at C . It is reqd. to find the locus of C .

Because A and B are fixed pts, therefore the \angle^s APB , AXB and AYB are of constant magnitudes. Therefore the ext. \angle XPB , being = the \angle PXB + the \angle XPB (Theor. 16 Obs) is constant; and therefore the ext. \angle ACB which is = the \angle CBY + the \angle CYB (Theor. 16, Obs.) is also constant. And this \angle ACB stands on a fixed line AB .

Hence the locus of C is the arc of a segment on the fixed chord AB , containing a constant angle = $\angle P + \angle X + \angle Y = \angle P + 2 \angle X$.

11. Let AHB be two circles intersecting at A and B . Let HAK be a fixed st. line drawn through A and terminated by the two circumferences, and let PAQ be any other st. line similarly drawn. Join HP and QK , and produce them to intersect at C . It is reqd. to find the locus of C .



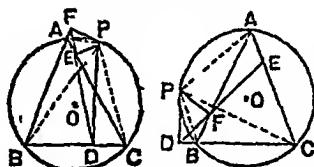
and ABQ be any st. line drawn through A and B . From P two st. lines PA , PB are drawn and produced to cut the other circle at Y and X . Join AY , BX intersecting at C . It is reqd. to find the locus of C .

Since the ext. \angle HPQ = the \angle HCK + the \angle PQC (Theor. 16, Obs) therefore the \angle HCK = the \angle HPQ - the \angle CQP . Because H , A and K are fixed pts., therefore the \angle^s AQK and APH which the arcs AK and AH subtend at the circumferences are of constant magnitudes.

Hence their difference is also constant. That is, the $\angle HCK$ is constant. Therefore the locus of C is the arc of a segment on the fixed chord HAK , containing an angle = the $\angle APH$ = the $\angle AQR$.

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1. Let P be circum-circle of $\triangle ABC$ and let PD , PF from P to BC , FD . Let it cut



any pt. on the the $\triangle ABC$, be perps drawn and AB . Join AC at E . Join

PE . It is reqd. to prove that PE is perp. to AC . Join AP , BP and CP .

Proof.—Because the \angle^s BFP and PDB are rt. angles, therefore the pts. P, D, B, F are concyclic (Converse, Theor. 40), and hence the $\angle FPB =$ the $\angle FDB$ (Theor. 39). Also the $\angle ACB =$ the $\angle APB$ (Theor. 39). In fig 1, or $\angle ACB = 180^\circ - \angle APB$ in fig. 2.

Fig 1.— \therefore the $\angle FPA =$ the $\angle FPB -$ the $\angle APB =$ the $\angle FDB -$ the $\angle ACB =$ the $\angle DEC$ (Theor. 16, Obs) = the $\angle AEF$ (Theor. 3).

Fig. 2—The $\angle AEF = \angle EDC + \angle ECD = \angle EPB + 180^\circ - \angle APB = 180^\circ - (\angle APB - \angle FPB) = 180^\circ - \angle APF$. $\therefore \angle AEF + \angle APF = 180^\circ$.

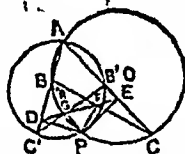
\therefore the pts. A, E, P, F are concyclic. Therefore the \angle^s AFP and AEP are supplementary (Theor. 40) in fig 1, or are equal (Theor. 39) in fig 2. But the $\angle AFP$ is a rt. angle, therefore the $\angle AEP$ is also a rt. angle. Hence PE is perp. to AC .

2. See fig. in Ex. 1. Let P be any such pt. that D, E, F , the feet of the perps. drawn from it on the sides of the given $\triangle ABC$ are collinear. It is reqd. to find the locus of P .

Because the \angle^s PEA and PFA are rt. angles, therefore the pts. F, A, E, P are concyclic (40), \therefore the $\angle APF =$ the $\angle AEF$ in fig. 1, or $= 180^\circ - \angle AEF$ in fig. 2, = the $\angle DEC$.

Again because the \angle^s PFB, PDB are rt. angles, therefore the pts. F, B, D, P are concyclic (Converse, Theor. 40). Therefore the \angle FPB = the \angle EDB (Theor. 39) \therefore in fig 1. \angle FPB - \angle FPA = \angle EDB - \angle DEC = ext \angle EDB - int. opp. \angle DEC = \angle ECD or \angle ACB (Theor. 16 Obs.), or, in fig 2, \angle FPB + \angle FPA = \angle EDB + \angle DEC = \angle^s EDC + DEC of the \triangle DEC = 180° - \angle ECD (or \angle ACB). That is, the \angle APB and the \angle ACB are equal, or supplementary. Hence, in either case, the pts. A, B, C and P are concyclic. Therefore the locus of P is the circum-circle of the \triangle ABC.

3. Let $\triangle ABC$ triangles with the circum-circles of meet again at P. PE, PF and PG



and $\triangle AB'C'$ be two common \angle A. Let these two triangles From P draw PD, perps. to AB, AC,

BC and $B'C'$ respectively. It is reqd. to prove that the pts D, G, F, E are collinear

Proof—Because PD, PF, PE are perps. drawn from P to the sides of the $\triangle ABC$, therefore the pts. D, F and E are collinear. [Prob V, page 212, Simson's line]. Again because PD, PG, PE are perps drawn from P on the sides of the $\triangle AB'C'$, therefore the pts. D, G and E are collinear [Prob. V, page 212] Hence the pts. D, G, F and E are collinear.

4 Let $\triangle ABC$ be in a given circle, and this circle. Let O of the $\triangle ABC$. Join PD, PE, PF perps respectively. Join



a triangle inscribed let P be any pt. on be the ortho-centre PO. From P draw to AB, AC and BC DF, then DF passes

through E (Prob. V, page 212), let it cut OP at G. It is reqd. to prove that OP is bisected by the st. line DEF at G.

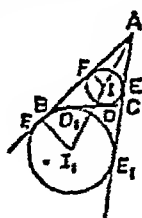
Let DP meet the circle again at M. Join MC. Produce DP to any pt. N making $PN = DM$. Determine Q as the circum-centre of the $\triangle ABC$ (Prob. 25), and draw QK, QL perps to AB, DN respectively. Then K and L

are the middle pts. of AB , MP respectively (Prob. 31). Join OC , then $OC = 2 QK$ (Ex. 10, page 209) $= 2DL = DN$ [because $DM + ML = PN + LP$, or $DL = LN$].

Proof—Because the \angle^s AEP and ADP are rt. angles therefore the pts. A, D, P and E are concyclic (Converse, Theor. 40), and hence the $\angle PAE = \angle PDE$ (Theor. 39). Again because the pts. A, C, P, M are concyclic, therefore the $\angle PMC = \angle PAC$ or PAE (Theor. 39) $= \angle PDE$. $\therefore DF$ and MC are parallel (Theor. 13). Also CH, PM are parallel, being perps. to the st. line AB . If CO cut DF at H , then the fig. $DHCM$ is a parallelogram. Therefore $HC = DM = PN$, and since $OC = DN$, $\therefore OH = DP$. Also OH is parallel to DP . Therefore the fig. $DOHP$ is a parallelogram (Theor. 20). Hence the diagonals DH, PO bisect one another at G (Cor. 3, Theor. 21). Hence OP is bisected by the st. line DEF at G .

Proof of the Equalities on Prop. VI, page 213.

(i) Because from A two tangents AE, AF are drawn to the inscribed circle, therefore $AE = AF$ (Cor. Theor. 47). Similarly it can be proved that $BD = BF$, and $CD = CE$.



A two tangents AE, AF are drawn to the inscribed circle, therefore $AE = AF$ (Cor. Theor. 47). Similarly it can be proved that $BD = BF$, and $CD = CE$.

Now $AB + BC + CA = AF + FB + BD + DC + CE + EA = 2 AE + 2 BD + 2 CD = 2 AE + 2 BC$. That is $2s = 2 AE + 2a$, or $2 AE = 2s - 2a$, $\therefore AE = s - a = AF$. Likewise it can be proved that $BD = BF = s - b$, and $CD = CE = s - c$.

(ii) Because from A two tangents AE_1, AF_1 are drawn to the escribed circle $\therefore AE_1 = AF_1$ (Cor. Theor. 47). Similarly it can be proved that $BF_1 = BD_1$ and $CE_1 = CD_1$.

$\therefore AB + BC + CA = AB + BD_1 + CD_1 + CA = AB + BF_1 + CE_1 + AC = AF_1 + AE_1 = 2 AE_1$. That is $2s = 2 AE_1$. Therefore $AE_1 = AF_1 = s$.

(iii) Because $AE_1 = s$ [proved in (ii)], therefore $CD_1 = CE_1 = AE_1 - AC = s - b$.

Again because $AF_1 = s$ [proved in (ii)], therefore $BD_1 = BF_1 = AF_1 - AB = s - c$.

(iv) Because $CD = s - c$ [proved in (i), and $BD_1 = s - c$ [proved in (iii)]; therefore $CD = BD_1$.

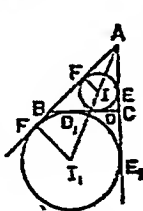
Again because $BD = s - b$ [proved in (i)], also $CD_1 = s - b$ [proved in (iii)], therefore $BD = CD_1$.

(v) Since $AE_1 = AF_1 = s$ [proved in (ii)], and $AE = s$ $AF = s - a$ [proved in (i)]; therefore $EE_1 = AE_1 - AE = s - (s - a) = a$; and $FF_1 = AF_1 - AF = s - (s - a) = a$. Hence $EE_1 = FF_1 = a$.

(vi) Area of the $\triangle ABC = \frac{1}{2} (a + b + c) r$ (Ex. 5, page 198) $= rs$, since $2s = a + b + c$.

Also its area $\frac{1}{2} (b + c - a) r_1$ (Ex. 6, page 198) $= [\frac{1}{2} (a + b + c) - a] r_1 = (s - a) r_1$.

(vii) If the $\angle C$ be a rt angle, then the figures $IDCE$ and $I_1 D_1 C E_1$ would be rectangles. $\therefore r = ID = CE = s - c$ [proved in (i)], and $r_1 = I_1 D_1 = CE_1 = s - b$ [proved in (iii)].



Proof of the properties on Prop VII, page 214

See fig in Ex. 11, page 189.

(i) Because IA bisects the $\angle BAC$ (Prob. 26), and $I_1 A$ also bisects the $\angle BAC$ (prob. 27), therefore the pts A, I and I_1 are collinear. Similarly it can be proved that the pts. B, I and I_2 , as well as the pts. C, I and I_3 are collinear.

(ii) Since $I_1 A$ and $I_2 A$ are the internal and external bisectors of the $\angle A$, therefore the $\angle I_1 A I_2$ is a rt. angle (Ex. 6 page 18). Similarly the $\angle I_2 A I_3$ is a rt. angle. Therefore the st. lines $I_2 A$ and $I_3 A$ are in one

st. line (Theor. 2). Hence the pts. I_2, A and I_3 are collinear. Similarly it can be proved that the pts. I_3, B and I_1 as well as the pts. I_1, C and I_2 are collinear.

(iii) Because AI_1 and AI_2 are the internal and external bisectors of $\angle A$, therefore $I_1 A$ is perp. to $I_2 A$ or $I_2 I_3$. Similarly it can be proved that $I_3 C$ is perp. to $I_1 I_2$ and that $I_2 B$ is perp. to $I_3 I_1$.

Therefore I is the orthocentre of the $\triangle I_1 I_2 I_3$, and ABC is the pedal triangle of the $\triangle I_1 I_2 I_3$. Therefore the $\triangle^s BI_1 C, CI_2 A, AI_3 B$ are equiangular to one another and to the $\triangle I_1 I_2 I_3$ [Prop. 11, Cor. (ii), page 208].

(iv) If the inscribed circle touch the sides BC, CA and AB at the pts. D, E and F , then the $\angle FDE = 90^\circ - \frac{A}{2}$ (Ex. 5, page 203). Also the $\angle BI_1 C = 90^\circ - \frac{A}{2}$ (Ex. 7, page 47). Therefore the $\angle FDE =$ the $\angle BI_1 C$. Similarly, it can be proved that the $\angle DEF =$ the $\angle AI_2 C$ and that the $\angle EFD =$ the $\angle AI_3 B$. Hence the $\triangle^s I_1 I_2 I_3$ and DEF are equiangular.

(v) Because I is the ortho-centre of the $\triangle I_1 I_2 I_3$ [proved in case (iii)], therefore of the four points I, I_1, I_2 and I_3 each is the ortho-centre of the triangle whose vertices are the other three (Ex. 4, page 209).

(vi) I is the ortho-centre of the $\triangle I_1 I_2 I_3$ [proved in case (iii)]; therefore the three circles which pass through two vertices of the $\triangle I_1 I_2 I_3$ and the pt. I are each equal to the circum-circle of the $\triangle I_1 I_2 I_3$ (Ex. 5, page 209). Hence the four circles, each of which passes through three of the pts. I, I_1, I_2, I_3 are all equal.

Page 215.

1. See fig. in Ex. 11, page 189; also in Ex. (i), page 213. (i) It has been proved in Ex. (ii), page 213, that $AE_1 = AF_1 = s$. Similarly it can be proved

that $BD_2 = s$, also $CD_3 = s$. $BD = s - b$ [proved in Ex. (i), page 213] and $CD_1 = s - b$ [proved in Ex. (iii), page 213]

Therefore $DD_2 = BD_2 - BD = s - (s - b) = b$ and $D_1D_3 = CD_3 - CD_1 = s - (s - b) = b$. Hence $DD_2 = D_1D_3 = b$.

(ii) $CD = BD_1$ [proved in Ex. iv, page 213] and $BD_1 = s - c$ [proved in Ex. (iii) page 213] Therefore $CD = BD_1 = s - c$. Now $DD_3 = CD_3 - CD = s - (s - c) = c$, and $D_1D_2 = BD_2 - BD_1 = s - s - c = -c$. Hence $DD_3 = D_1D_2 = c$.

(iii) $D_2D_3 = DD_2 + DD_3 = b + c$ [from (i), and (ii)].

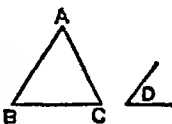
(iv) $DD_1 = DD_2 \sim D_1D_2 = b \sim c$.

2. See fig in Ex 1. I is the ortho centre of the $\triangle I_1I_2I_3$, as well as the in centre of its pedal triangle ABC . And the vertices I_1, I_2 and I_3 of the $\triangle I_1I_2I_3$ are the centres of the escribed circles of the $\triangle ABC$. Therefore the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.

3. See fig. in Ex 1, page 211.—Let X be the given angle and BC the given base. Let ABC be any triangle on the given base BC having the vertical $\angle A = \angle X$. Produce AB, AC to pts. D and E , and bisect the \angle s DBC, ECB by the st. lines BI_1 and CI_1 meeting at I_1 . It is reqd to find the locus of I_1 .

Since the $\angle BI_1C = 90^\circ - \frac{A}{2}$ (Ex. 7, p. 47), and the $\angle A$ is constant, $\therefore \angle BI_1C$ is also constant. \therefore the locus of I_1 is the arc of a segment on the fixed chord BC containing an angle $= 90^\circ - \frac{A}{2}$.

4. Let BC be the given base. Let D the given angle. $\angle A = \angle D$. It is the circum-centre of



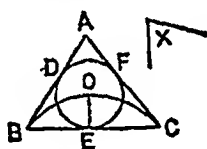
the given base, and Let ABC be a triangle on the base BC , having its vert. $\angle = \angle D$. It is reqd to prove that the $\triangle ABC$ is fixed.

Since the vertical $\angle BAC$ is constant, and the base BC is fixed, \therefore the locus of vertex A is the arc of a segment on BC as its chord, containing an angle $= \angle D$ (Prob. 24). But this arc circumscribes the $\triangle ABC$, \therefore the circum-circle is fixed, and hence its centre is also fixed.

5. See fig. in Ex. 11, page 189 Let ABC be a triangle on the given base BC , and having its vertical $\angle BAC =$ the given vertical angle. Let I_2 be the centre of the escribed circle touching the side AC . It is reqd. to find the locus of I_2 .

Because the $\angle^s I_2BI_1$ and I_1AI_2 are rt. \angle^s , \therefore the pts. I_1, B, A and I_2 are concyclic (Theor. 39 converse) $\therefore \angle$ the $BI_2I_1 =$ the $\angle BAI_2$ (Theor. 39) $= \frac{1}{2}A =$ constant. \therefore the locus of I_2 is the arc of a segment on BC as a chord containing an angle $= \frac{1}{2}A$.

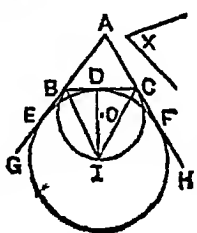
6. Let BC be the given vertical point of contact with incircle. It is reqd. to construct the triangle.



the given base, X angle, and E the the base BC of the to construct the

The locus of the in-centre O is the arc of a segment on BC as chord containing an angle $= 90^\circ + \frac{1}{2}X$ (Prop. IV, p. 210). From E draw EO perp. to BC meeting this arc at O . Then O is the in-centre of the triangle and OE the in radius. With centre O and radius OE draw a circle. From B, C draw tangents to this circle (Prob. 22) and let the tangents meet at the pt. A . Then ABC is the reqd. triangle.

7. Let BC be the given vertical point of contact of with the base BC . It is reqd. to construct the triangle.

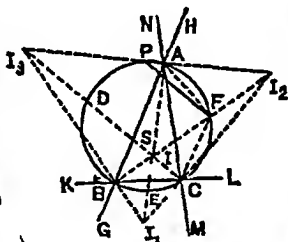


the given base, X angle, and D the the escribed circle It is reqd. to cons

The locus of the excentre I is the arc of a segment on BC as chord containing an angle $= 90^\circ - \frac{X}{2}$ (Ex. 1, P.

211). Draw this arc. From D draw DI perp to BC meeting this arc at I . Then I is the centre and ID the radius of the escribed circle. With centre I and radius ID draw a circle, from B and C draw tangents to this circle, and produce them to meet at the pt A . Then ABC is the reqd. triangle

8 Let I be inscribed circle I_1 the centres circles of the Δ circum-circle of II_1, II_2, II_3 at pt. I respectively. It is reqd. to show that E, F and D are the mid. pts. of II_1, II_2, II_3 .



the centre of the and I_1, I_2 and of the escribed ABC , and let the the ΔABC cut E, F and D resqd. to show are the mid. pts.

Join AF CF . The $\angle AFC = 180^\circ - B$ (Theor. 40), and the $\angle AI_2C = 90^\circ - \frac{1}{2}B$ (Ex. 7, p 47). \therefore the $\angle AFC = 2 \angle AI_2C$. Again because the $\angle^s IAI_2$ and ICI_2 are rt angles, therefore the circle on diameter II_2 passes through A and C (Ex. I, page 165). \therefore the centre of this circle lies on II_2 and since F is a pt. on II_2 , such that $\angle AFC = 2 \angle AI_2C$, F must be the centre of this circle. Hence II_2 is bisected at F . Similarly it can be proved that II_2 and II_3 are bisected at E and D .

9. See fig in Ex. 8 —Let I_2, I_3 be the centres of the escribed circles which touch the sides AC and AB of the ΔABC . It is reqd. to prove that the pts. B, C, I_2 and I_3 all lie on a circle whose centre is on the circum-circle of the ΔABC .

Proof—Because the $\angle^s I_2BI_3$ and I_2CI_3 are rt. angles, therefore the pts I_2, C, B and I_3 lie on a circle whose diameter is I_2I_3 (Ex. 1, page 165). Bisect

$I_2 I_3$ at P ; then P is the centre of this circle. Join FP . Because $I I_2$ is bisected at F (proved in Ex. 8), and $I_3 I_2$ at P , therefore FP is parallel to $I_3 I_1$ (Ex. 2 page 64), hence the ext. $\angle APF =$ the int. $\angle I_2 I_3 C$ (Theor. 14. Again because the $\angle^s I A I_3$ and $I B I_3$ are rt. angles, therefore the pts. I, A, I_3 and B are concyclic (converse, Theor. 40) \therefore the $\angle I_3 I_1 =$ the $\angle A B I$ (Theor. 39). Therefore the $\angle APF =$ the $\angle A B I$ or the $\angle A B F$; and since they stand on the same line AF the pts. A, P, B , are concyclic (Converse, Theor. 39) But the pt. F lies on the circum-circle of the $\triangle ABC$ which passes through A and B . Hence the pt. P also lies on the circum-circle of the $\triangle ABC$.

10. See fig in Ex 1, p. 213.

Let A, B, C , be the three given points. It is reqd. to draw with A, B and C as centres, three circles which may touch one another two by two, also to show how many solutions there are.

(1) Let the inscribed circle of the $\triangle ABC$ touch the sides BC, CA and AB at D, E and F respectively. Then $AE=AF, BD=BF$ and $CD=CE$ (Ex. 1, p. 213). \therefore the circles described with centres A, B and C and radii AF, BD and CE respectively, will touch each other externally two by two. (ii) Let the escribed circle with I_1 as centre touch the sides AB, BC and CA at the pts. F_1, D_1 and E_1 respectively. Then $AE_1=AF_1, BD_1=BF_1$ and $CD_1=CE_1$ (Ex. 11 and 12, p. 213), \therefore the circles described with centres A, B and C and radii AE_1, CD_1 and BF_1 will touch each other two by two.

If the escribed circle with I_2 and I_3 as centres touch the sides BC, CA, AB , at the pts. D_2, E_2, F_2 and D_3, E_3 and F_3 respectively; then it can similarly be shown that the circles described with A, B and C as centres and radii AE_2, CD_2 , and BF_2 , as also the circles described with centres A, B, C and radii AE_3, CD_3, BF_3 will touch each other two by two. Hence it is clear that there are four solutions of this problem.

11. See fig in Ex 1—

Let I_1, I_2 and I_3 be the centres of the three escribed circles. It is reqd. to construct the triangle.

Analysis:—Let ABC be such a triangle. Join $I_1 I_2, I_2 I_3, I_3 I_1$ and from I_1, I_2 and I_3 draw perps. to the opp. sides intersecting at I_1 . Since I is the orthocentre of the $\triangle I_2 I_3 I_1$, it is the incentre of the $\triangle ABC$; the pts. A, I, I_1 , are collinear; so are the pts. B, I, I_2 , and C, I, I_3 , (Exs. (v) and (1), p. 214). $\therefore A, B, C$ are the feet of the perps. drawn from I_1, I_2, I_3 . Hence we get the following construction.

Construction:—Join $I_1 I_2, I_2 I_3, I_3 I_1$, from I_1, I_2, I_3 drop perps. to the opp sides, and let A, B, C , be the feet of these perps. Join AB, BC, CA . Then ABC is the reqd triangle.

12 See fig. in Ex. I.

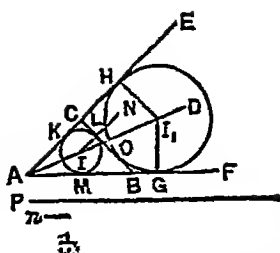
Let I be the centre of the inscribed circle, and I_3, I_2 the centres of two escribed circles. It is reqd. to construct the triangle.

Analysis—Let ABC be such a triangle. Then A, I, I_1 are collinear, so are B, I, I_2 . Also if I_3 be the third ex-centre, then I_2, A, I_3 , are collinear, so are I_1, B, I_3 ; and I, C, I_3 . \therefore lines $IC, I_1 B, I_2 A$ drawn from the vertices of the $\triangle I I_2 I_3$, pass through I_3 . But I_3 is the ortho-centre of this \triangle .

$\therefore IC, I_1 B, I_2 A$ are perps. drawn from the vertices of this \triangle to the opp sides, and C, B, A are the feet of these perps. Hence we have the following construction.

Construction—Join $I I_1, I_1 I_2, I_2 I_3$. From I, I_1, I_2 , draw perps to the opp. sides, and let C, A, B be the feet of these perps. Join AB, BC and CA . Then ABC is the reqd. triangle.

13. Let $\angle EAF$ be
cal angle, P the
and r the radius
circle. It is reqd.
triangle.



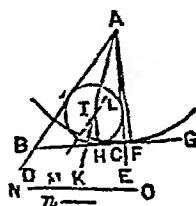
the given verti-
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Analysis:—Let ABC be such a triangle, and let I, I_1 be the centres of the inscribed circle and the escribed circle touching the side BC . Then the points A, I and I_1 , are collinear. Through I draw a st. line IL parallel to AE ; then its distance from $AE = r$. From I_1 draw I_1H \perp to AE, AF , then $AH = AG = \frac{1}{2} P$. (Ex. II, p 313). BC is the transverse common tangent to the inscribed and escribed circles. Hence we get the following construction

Constructions:—Bisect the $\angle EAF$ by AD . Draw a st line IL parallel to AE and at a distance $= r$ from it, cutting AD at I . From AE cut off AH equal to $\frac{1}{2} P$. At H draw HI_1 \perp to AE cutting AD at I_1 . With I, I_1 as centres and radii $= r, I, H$ respectively draw two circles. Then these circles will touch both AE and AF .

Draw a transverse common tangent to these two circles intersecting AE and AF at the pts. C and B respectively. Then ABC is the reqd. triangle.

14. Let $\angle DAE$ be
angle, NO the length
the vertex to the
radius of the in-
is reqd. to construct



the given vertical
of the perp. from
base, and r the
scribed circle. It
the triangle.

Analysis:—Let ABC be such a triangle, and I be the centre of its inscribed circle. Join AI , then AI bisects the $\angle DAE$. Through I draw a st. line MIL parallel to AB ; then it is at a distance $= r$ from AD .

From A draw AF perp. to BC , then $AF = NO$. With centres I and A and radii $= r$ and NO respectively draw two circles. Since the former is the inscribed circle of the triangle, and since $\angle AFB$ is a rt. angle, BC is the direct common tangent to these two circles. Thus we get the following constructions

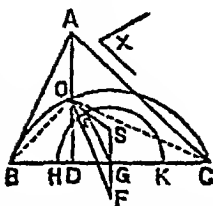
Constructions — Bisect the $\angle DAE$ by the st. line AK , and draw a st. line ML parallel to AD at a distance $= r$ from it, intersecting AK at I. With centres I and A, and radii $= r$ and NO respectively draw two circles. Draw a direct common tangent to these two circles, and let it cut AD, AE at B, C. Then ABC is the required triangle

15. See fig in Ex. 8.

Let ABC be a triangle, and I the centre of the inscribed circle. It is required to prove that the centres of the circles circumscribed about the triangles BIC, CIA and AIB lie on the circumference of the circum-circle of the triangle ABC .

Let I_1, I_2, I_3 be the centres of the three escribed circles. Join AI_1, BI_2, CI_3 , then each of them passes through I. Join $I_1 I_2, I_1 I_3, I_2 I_3$; then C, A, B lie on these lines. Let the circle about the $\triangle ABC$ cut II_1, II_2, II_3 at the pts. E, F, D respectively. Join AF and CF. It has already been proved in Ex. 8, that the fig. $AIC I_2$ is concyclic, and that F is the centre of the circumscribing circle. Hence the centre F of the circle circumscribed about the triangle CIA lies on the circum-circle of the $\triangle ABC$. Similarly it can be proved that E and D are the centres of the circles circumscribed about the $\triangle BIC$ and AIB ; and they lie on the circum-circle of the $\triangle ABC$.

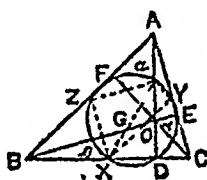
1. Let BC be X the given vertical angle ABC to be a triangle having its vertical angle A . It is reqd. to find centre of the nine-



the given base and vertical angle. Suppose $\angle A = \angle X$. the locus of the points circle.

Since the base BC and the vertical angle is given, the circum-circle of the $\triangle ABC$ is fixed (Prob. 24). \therefore circum-radius is constant \therefore the radius of the nine-points circle $= \frac{1}{2}$ circum-radius = constant. [Property (11), page 217] And since the nine-points circle always passes through G the mid pt. of BC , its centre is always at a distance $= \frac{1}{2}$ circum-radius, from the pt. G . \therefore its locus is arc HEK of the circle whose centre is G , the mid. pt. of BC , and radius $= \frac{1}{2}$ circum-radius.

2. Let ABC be a triangle. O be its ortho-centre. It is reqd. to prove points circle of the the nine-points circle $\triangle^s AOB, BOC, COA$.



triangle, and let O be its ortho-centre. Join AO, BO, CO . that the nine-points circle of each of the $\triangle^s AOB, BOC, COA$.

The nine-points circle of the $\triangle ABC$ passes through the mid. pts. of AB, AO, BO (Theor. VIII, page 216), i.e., through the three mid. pts. of the sides of the $\triangle AOB$. Since one and only one circle can pass through three points not in one st. line (Theor. 32), and since the nine-points circle of a triangle passes through the three mid. pts. of its sides, \therefore the nine-points circle of the $\triangle ABC$ must be the nine-points circle of the $\triangle AOB$.

Similarly it can be shown that it is also the nine-points circle of each of the $\triangle^s BOC$ and COA .

3. See fig in Ex. II, page 189 Let I, I_1, I_2, I_3 be the centres of the inscribed and the escribed circles of a $\triangle ABC$.

It is reqd to prove that the circle circumscribed about the $\triangle ABC$ is the nine-points circle of each of the \triangle^s $II_1 I_2$, $II_2 I_3$, $I I_3 I_1$ and $I_1 I_2 I_3$.

∴ From Theor. VIII on page 216 and Theor. 32 we know that in a triangle the circle passing through the feet of the perps. drawn from its vertices to the opp. sides, is the nine-points circle of the \triangle .

It can be easily seen that in each of the \triangle^s $I I_1 I_2$, $II_2 I_3$, $II_3 I_1$, $I_1 I_2 I_3$, A, B, C are the feet of the perps. drawn from the vertices to the opp. sides. Hence the circle through A, B, C , is the nine-points circle of each of the above triangles.

4. It is reqd to prove that all triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

Since all the \triangle^s have the same circum-circle, their common circum-centre is a fixed pt., and common circum-radius is of constant length. Also their common orthocentre is a fixed pt.

∴ the centre of the nine-points circle, which is the mid. pt. of the st line joining the orthocentre and the circum-centre, is a fixed pt. also. And the radius of the nine-points circle = half the common circum-radius = constant.

Hence, all the triangles have the same nine-points circle.

5. See fig in Ex. 2 (2), page 209. Let ABC be a triangle having its base, BC = the given base and the $\angle ABC$ = the given vertical angle. Let DEF be its pedal triangle. It is reqd. to prove that one angle and one side of the pedal \triangle are constant.

Join AD, BE, CF intersecting at O . Since FO bisects the $\angle EFD$, and EO bisects the $\angle FED$, ∴ the $\angle FOE = 90^\circ + \frac{1}{2}$ the $\angle FDE$. (Ex. 6, page 47).

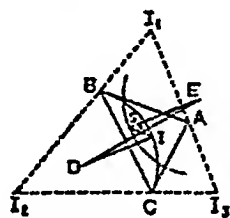
In the quadl. $AFOE$, since $\angle AFO + \angle AEO = 90^\circ + 90^\circ = 180^\circ$, \therefore the other two $\angle^s FAE + FOE = 180^\circ$, or $\angle FOE = 180^\circ - \angle FAE = 180^\circ - \angle A$. $\therefore 90^\circ + \frac{1}{2}$ the $\angle FDE = 180^\circ - \angle A$, $\therefore \frac{1}{2}$ the $\angle FDE = 90^\circ - \angle A = \text{constant}$, since $\angle A$ is given to be constant. $\therefore \angle FDE$ is also constant.

Again because the base and the vertical angle of the $\triangle ABC$ are given, \therefore its circum-circle is fixed (Prob. 24). \therefore its circum-radius is of constant length. \therefore radius of the nine-points circle, which is $=$ half the circum-radius, is also of constant length; i. e., the nine-points circles of all the \triangle^s whose base $=$ the given base, and vertical $\angle =$ the given vertical \angle , are all equal to one another.

Now EF is a chord of the nine-points circle, and it subtends a constant angle FDE at circumference. \therefore it is of constant length (Theors. 42 and 45).

Thus one angle FDE , and one side EF of the pedal $\triangle DEF$ are constant

6. Let ABC be given base BC , and angle $BAC =$ the g l . Let N, I, I_1 , circum-centre, in-centre, respectively. It is



a triangle on the having its vertical an-
given vertical an-
 I_2, I_3 be its cir-
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locus of the circum-centre of the $\triangle I_1 I_2 I_3$. Join $I_1 I_2, I_2 I_3, I_3 I_1$. Then A, B, C lie on these lines. Also I is the ortho-centre of the $\triangle I_1 I_2 I_3$ [Property (v), page 214]. And since A, B, C are the feet of the perps. drawn from the vertices I_2, I_3, I_1 on opp. sides; \therefore the circle through A, B, C , i. e. the circum-circle of the $\triangle ABC$ is the nine-points circle of the $\triangle I_1 I_2 I_3$. $\therefore N$ is the centre of the nine-points circle of the $\triangle I_1 I_2 I_3$. Join IN and produce it to S making $NS = IN$. Then S is the circumcentre of the $\triangle I_1 I_2 I_3$ [Property (i), page 217].

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